Levels of Knowledge and Belief

Computational Social Choice Seminar

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Introduction and Motivation

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Introduction and Motivation

Motivating Questions/Issues:

- How do states of knowledge influence decisions in *game situations*?

- Can we *realize* any state of knowledge?

- What is a *state* in an epistemic model?

- Is an epistemic model *common knowledge* among the agents?

States of Knowledge in Games

R. Parikh. *Levels of knowledge, games and group action*. Research in Economics 57, pp. 267 - 281 (2003).

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$$C_{p,m}c$$

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$$K_p c, \neg K_m K_p c$$

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What about other levels of knowledge?

R. Parikh and P. Krasucki. *Levels of knowledge in distributed computing*. Sadhana-Proceedings of the Indian Academy of Science 17 (1992).

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What about in *game situations*?
Answer: a *description* of the first-order and higher-order information of the players

R. Fagin, J. Halpern and M. Vardi. *Model theoretic analysis of knowledge*. Journal of the ACM 91 (1991).

Is an Epistemic Model "Common Knowledge"?

"The implicit assumption that the information partitions...are themselves common knowledge...constitutes no loss of generality... the assertion that each individual 'knows' the knowledge operators of all individual has no real substance; it is part of the framework."

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"it is an informal but *meaningful* meta-assumption....It is not trivial at all to assume it is "common knowledge" which partition every player has."

A. Heifetz. *How canonical is the canonical model? A comment on Aumann's interactive epistemology*. International Journal of Game Theory (1999).

A General Question

How many levels/states of knowledge (beliefs) are there?

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It depends on how you count:

- Parikh and Krasucki: Countably many *levels* of knowledge
- Parikh and EP: Uncountably many levels of belief
- Hart, Heiftetz and Samet: Uncountably many *states* of knowledge

Levels of Knowledge

Fix a set of agents $\mathcal{A} = \{1, \ldots, n\}$.

$\Sigma_K = \{K_1, \ldots, K_n\}$ and $\Sigma_C = \{C_U\}_{U \subseteq \mathcal{A}}$

Level of Knowledge: $Lev_{\mathcal{M}}(p, s) = \{x \in \Sigma^* \mid \mathcal{M}, s \models xp\}$
(where $\Sigma = \Sigma_K$ or $\Sigma = \Sigma_C$).

[If Σ is a finite set, then Σ^* is the set of finite strings over Σ]
[Recall the definition of truth in a Kripke structure]

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Consider the sets:

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- $L_1 = \{K_1, K_3, K_1K_2K_3\}$ and $L_2 = \{K_1, K_2, K_3, K_1K_2K_3\}$

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- $L_1 = \{K_1, K_3, K_1K_2K_3\}$ and $L_2 = \{K_1, K_2, K_3, K_1K_2K_3\}$ (*same level of knowledge*)

Levels of Knowledge: Preliminaries

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1. $x \leq x$ and $\epsilon \leq x$ for all $x \in \Sigma^*$
2. $x \leq y$ if there exists $x', x'', y', y'', (y, y'' \neq \epsilon)$ such that $x = x'x''$, $y = y'y''$ and $x' \leq y'$, $x'' \leq y''$.

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Example:
$aba \leq aaba$
$aba \leq abca$
$aba \not\leq aabb$

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A set $\{a_1, a_2, \ldots\}$ of incomparable elements is a well-founded partial order but not a WPO.

Well-Partial Orders

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Theorem (Higman). If Σ is finite, then $(\Sigma^*, \leq)$ is a WPO

G. Higman. *Ordering by divisibility in abstract algebras*. Proc. London Math. Soc. 3 (1952).

D. de Jongh and R. Parikh. *Well-Partial Orderings and Hierarchies*. Proc. of the Koninklijke Nederlandse Akademie van Wetenschappen 80 (1977).

WPO and Downward Closed Sets

Given $(X, \preceq)$ a set $A \subseteq X$ is **downward closed** iff $x \in A$ implies for all $y \preceq x$, $y \in A$.

Theorem. (Parikh & Krasucki) If Σ is finite, then there are only countably many $\leq$-downward closed subsets of Σ^* and all of them are *regular*.

Levels of Knowledge

Theorem. Consider the alphabet $\Sigma_C = \{C_U\}_{U \subseteq \mathcal{A}}$. For all strings $x, y \in \Sigma_C^*$, if $x \preceq y$ then for all pointed models $\mathcal{M}, s$, if $\mathcal{M}, s \models yP$ then $\mathcal{M}, s \models xP$.

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Corollary 1. Every level of knowledge is a downward closed set.
Corollary 2. There are only countably many levels of knowledge.

Realizing Levels of Knowledge

Theorem. (R. Parikh and EP) Suppose that L is a downward closed subset of Σ_K^*, then there is a finite Kripke model $\mathcal{M}$ and state s such that $\mathcal{M}, s \models xP$ iff $x \in L$. (i.e., $L = Lev_{\mathcal{M}}(p, s)$).

States of Knowledge

S. hart, A. Heifetz and D. Samet. *"Knowing Whether," "Knowing That," and The Cardinality of State Spaces*. Journal of Economic Theory 70 (1996).

States of Knowledge

Let W be a set of states and fix an event $X \subseteq W$.

Consider a sequence of finite boolean algebras $\mathcal{B}_0, \mathcal{B}_1, \mathcal{B}_2, \ldots$ defined as follows:
$\mathcal{B}_0 = \{\emptyset, X, \neg X, \Omega\}$
$\mathcal{B}_n = \mathcal{B}_{n-1} \cup \{K_i E \mid E \in \mathcal{B}_{n-1}, i \in \mathcal{A}\}$

The events $\mathcal{B} = \cup_{i=1,2,\ldots}\mathcal{B}_i$ are said to be **generated by** X.

States of Knowledge

Definition. Two states w, w' are **separated** by X if there exists an event E which is generated by X such that $w \in E$ and $w' \in \neg E$.

Question: How many states can be in an information structure (W, Π_1, Π_2) such that an event X separates any two of them?

States of Knowledge

Consider a K-list $(E_1, E_2, E_3, \ldots)$ of events generated by X.

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$(X, K_1X, \neg K_2K_1X, \neg K_1\neg K_2K_1X, K_2\neg K_1\neg K_2K_1X)$ is inconsistent.

Knowing Whether

Let $J_i E \ := \ K_i E \vee K_i \neg E$.

Lemma. Every J-list is consistent.

Theorem. (Hart, Heifetz and Samet) There exists an information structure (W, Π_1, Π_2) and an event $X \subseteq W$ such that all the states in W are separated by X and W has the cardinality of the continuum.

S. hart, A. Heifetz and D. Samet. *"Knowing Whether,"*, *"Knowing That," and The Cardinality of State Spaces*. Journal of Economic Theory 70 (1996).

What about beliefs?

In Aumann/Kripke structures belief operators are just like knowledge operators except we replace the truth axiom/property ($K\varphi \rightarrow \varphi$) with a consistency property ($\neg B\bot$).

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What about probabilistic beliefs?

Bayesian Structures

- Let W be a set of worlds and $\Delta(W)$ be the set of probability distributions over W.
- We are interested in functions $p : W \rightarrow \Delta(W)$.
- The basic intuition is that for each state $w \in W$, $p(w) \in \Delta(W)$ is a probability function over W.
- So, $p(w)(v)$ is the probability the agent assigns to state v in state w. To ease notation we write p_w for $p(w)$.

Definition

The pair $\langle W, p \rangle$ is called a **Bayesian frame**, where $W \neq \emptyset$ is any set, and $p : W \rightarrow \Delta(W)$ is a function such that

$$\text{if } p_w(v) > 0 \text{ then } p_w = p_v$$

Given a Bayesian frame $\mathcal{F} = \langle W, p \rangle$ and a set of states S, an **Bayesian model based on S** is a triple $\langle W, p, \sigma \rangle$, where $\sigma : W \rightarrow S$.

Definition

For each $r \in [0,1]$ define $B^r : 2^W \rightarrow 2^W$ as follows

$$B^r(E) = \{w \mid p_w(E) \geq r\}$$

Observation: We can define a possibility model from a Bayesian model as follows. Let $\langle W, p, \sigma \rangle$ be a Bayesian model on a state space S. We define a possibility model $\langle W, P, \sigma \rangle$ base on S as follows: define $\mathcal{P} : W \to 2^W$ by

$$\mathcal{P}(w) = \{v \mid \pi_w(v) > 0\}$$

It is easy to see that $\mathcal{P}$ is serial, transitive and Euclidean.

Example

Let $N = \{A, B\}$, $\Omega = \{\alpha, \beta, \gamma\}$ and define π_A and π_B as follows:

π_A:

	α	β	γ
α	1/2	1/2	0
β	1/2	1/2	0
γ	0	0	1

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Let $E = \{\alpha, \gamma\}$ and β be the *actual state*

Each player assigns probability $1/2$ to E.

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and so on...

Hierarchies of Beliefs (Belief Frames)

Fix a state $w \in \Omega$ and a Belief Frame.

- **Ground Facts** $\Phi_0 = \{E \mid w \in E \text{ and } E \text{ a ground event}\}$

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Thus each state in a belief models corresponds to the following infinite hierarchy of beliefs

$$(\Phi_0, (\Phi_i^1, \Phi_i^2, \ldots)_{i \in \mathcal{A}})$$

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Second Order Belief: $\mu^2_{i,\omega} \in \Delta(S \times [\Delta(S)]^{n-1})$ such that for all $s \in S$

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Given any state ω we have the following infinite hierarchy of beliefs

$$(\sigma(\omega), (\mu^1_{i,\omega}, \mu^2_{i,\omega}, \ldots)_{i \in \mathcal{A}})$$

Refining the Questions

- For any given set S of external states we can use a Bayesian model or a type space on S to provide consistent representations of the players' beliefs.

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- For any given set S of external states we can use a Bayesian model or a type space on S to provide consistent representations of the players' beliefs.
- Every state in a belief model or type space induces an infinite hierarchy of beliefs, but *not all consistent infinite hierarchies are in any finite model*. It is not obvious that even in an infinite model that all consistent hierarchies of beliefs can be represented.
- Which space is the "correct" one to work with?

Is there a universal (belief) space?

A **universal (belief) space** is a types space to which every type space (on the same space of states of nature and same set of agents) can be mapped, preferably in a unique way, by a map that preserves the structure of the type space.

If such a space exists, then the any analysis of a game could be carried out in this space without the risk of missing any "relevant" states of affairs.

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The existence of a such a space depends on the topological and/or measure theoretic assumptions being made about the underlying state space S.

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First shown by Mertens and Zamir (1985)

The problem is to define the set of all infinite hierarchies of beliefs satisfying the same consistency properties (coherency and common knowledge of coherency) as that of hierarchies obtained at some state in a type space.

Kolomogorov Extension Theorem

Why do we care?

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> It is not enough [...] that Ann should consider each of Bob's strategies possible. Rather, she considers possible both every strategy that Bob might play and every type that Bob might be. (Likewise, Bob considers possible both every strategy that Ann might play and every type that Ann might be.)

Brandenburger, Friedenburg and Keisler. *Admissibility in Games*. Econometrica 2009.

Overview of the Literature

- Existence proofs (under various topological assumptions): [Armbruster and Boge, 1979], [Mertens and Zamir, 1985], [Brandenburger and Dekel, 1993], [Heifetz, 1993], [Heifetz and Samet, 1998], [Battigalli and Siniscalchi, 1999], [Meier, 2002], [Salonen, 2003]

- Impossibility Result: [Brandenburger and Kesiler, 2004], [Meier, 2005]

Overview of the Literature

- Knowledge structures: [Fagin, Halpern and Vardi, 1991], [Heifetz and Samet, 1998], [Fagin, Geanokoplos, Halpern and Vardi , 1999]
- Relevant surveys: [Aumann, 1999], [Bonanno and Battigalli, 1999], [Brandenburger, 2002], [Brandenburger, 2005]
- Logic of type spaces: [Heifetz and Mongin, 2001], [Meier, 2001]

Brandenburger and Dekel

Assumption: Assume there are only two agents: i, j. Let the state space S be a Polish space (complete separable metric). For any metric space X assume that $\Delta(X)$ is endowed with the weak topology.

The proof proceeds as follows

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2. Define a notion of *coherency* such that if an individual's type is assumed to be coherent then it induces a belief over the types of the other individuals.

3. If common knowledge (in the sense of assigning probability 1) of coherency is assumed, then the set of beliefs is closed.

Step 1.

$$\begin{array}{rcl} X_0 & = & S \\ X_1 & = & X_0 \times \Delta(X_0) \\ & \vdots & \\ X_n & = & X_{n-1} \times \Delta(X_{n-1}) \\ & \vdots & \end{array}$$

A *type* t^i of i is an infinite sequence $t^i = (\delta_1^i, \delta_2^i, \ldots) \in \Pi_{n=0}^{\infty}\Delta(X_n)$

Let $T_0 = \Pi_{n=0}^{\infty}\Delta(X_n)$.

Step 2.

Coherent: A type $t = (\delta_1, \delta_2, \ldots) \in T_0$ is *coherent* if for every $n \geq 2$, $marg_{X_{n-2}} \delta_n = \delta_{n-1}$.

Coherency simply says that different levels of beliefs of an individual do not contradict one another. Let T_1 be the set of all coherent types.

Proposition There is a homeomorphism $f : T_1 \rightarrow \Delta(S \times T_0)$.

This is essentially *Kolmogorov's Existence Theorem*.

Note that the marginal probability assigned by $f(\delta_1, \delta_2, \ldots)$ to a given event in X_{n-1} is equal to the probability that δ_n assigns to that same event.

Step 3.

We now impose "common knowledge" of coherency:

For $k \geq 2$ define

$$T_k = \{t \in T_1 : f(t)(S \times T_{k-1}) = 1\}$$

Let $T = \cap_{k=1}^{\infty} T_k$

This set T is the set we are looking for: the universal type space.

Proposition There is a homeomorphism $g : T \rightarrow \Delta(S \times T)$

Conclusions: Returning to the Motivating Questions

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- Can we *realize* any state of knowledge?
- What is a *state* in an epistemic model?
- Is an epistemic model *common knowledge* among the agents?

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  It depends...

Thank You!