Eric Pacuit
Samer Salame

Majority Logic: Axiomatization and Completeness

Abstract. Graded modal logic, as presented in [5], extends propositional modal systems with a set of modal operators $\Diamond_n$ ($n \in \mathbb{N}$) that express "there are more than n accessible worlds such that...". We extend* **GML** with a modal operator W that can express "there are more than or equal to half of the accessible worlds such that...". The semantics of W is straightforward provided there are only finitely many accessible worlds; however if there are infinitely many accessible worlds the situation becomes much more complex. In order to deal with such situations, we introduce a majority space. A majority space is a set W together with a collection of subsets of W intended to be the weak majority (more than or equal to half) subsets of W. We then extend a standard Kripke structure with a function that assigns a majority space over the set of accessible states to each state. Given this extended Kripke semantics, majority logic is proved sound and complete.

Keywords: Modal Logic, Graded Modal Logic, Majority, Ultrafilters, Majority of infinite sets, Extended Kripke models.

1. Introduction

The language of modal logic has long been used to model intensional notions such as knowledge, belief and obligation. In this paper we present a new modal logic which models an agent's ability to reason about majorities. The concept of majority often plays an important role when an agent is faced with a decision in a social situation. For example, think of dinner with a group of friends. Chances are that many of the decisions, such as choice of restaurant, appetizers or wine, were based on the will of the majority. An extended example which illustrates this point is found in the next section. Of course, the concept of majority is integral to many voting systems. With these intuitions in mind, we propose a logic, **MJL**, in which the concept of majority is axiomatized.

Given a formula α, the language of propositional modal logic can express "α is true in *all* accessible worlds" ($\Box\alpha$), and "α is true in *at least one* accessible world" ($\Diamond\alpha$). But suppose that we want to express that α is true in at least *three* accessible worlds or that α is true in a *majority (more than half)* of the accessible worlds. The language of propositional modal logic cannot express such statements. The logic **MJL** presented in this paper will

*An preliminary version of this paper was presented at KR 2004 in Vancouver, Canada and appeared in the proceedings ([12].)

Studia Logica **0**: 1–36, 2006.

use modal operators that can specify exactly how many accessible worlds are of interest.

To start with, we add the graded modalities first discussed in [7, 8]. For each $n \in \mathbb{N}$, the formula $\Diamond_n \alpha$ is intended to mean α is true in strictly more than n accessible world, and so its dual $\Box_n \alpha$ is intended to mean $\neg\alpha$ is true in less than or equal to n accessible worlds. We may call $\Diamond_n \alpha$ an *at least* formula, since $\Diamond_n \alpha$ will be true precisely when α is true in *at least* $n+1$ accessible worlds. Similarly we may call $\Box_n \alpha$ *all but* formulas, since $\Box_n \alpha$ will be true precisely when α is false in *all but* n accessible worlds. For simplicity we write $\Diamond\alpha$ instead of $\Diamond_0 \alpha$ and $\Box\alpha$ instead of $\Box_0 \alpha$. For instance, if the formula $\Box_k \bot$ is true at some world w, then w has at most k accessible worlds.

We then extend the graded modal language with a new modal operator W, where $W\alpha$ is intended to mean α is true in more than or equal to half of the accessible worlds. Hence its dual, $M\alpha$ will mean α is true in more than half of the accessible worlds. Here M represents strict Majority and W represents Weak majority. In what follows, when we use "majority", we mean weak majority (i.e. more than or equal to 50%).

Before proceeding we should check that we are in fact gaining expressive power with the new modal operators. To see this, note that **MJL** does not obey bisimilation. We can easily find two bisimilar Kripke models where in one of them we have $W\alpha$ is true at some state s and in the other $W\alpha$ may not be true at a bisimilar state. It follows that the operator W cannot be defined from the standard modal operators ($\Box$ and $\Diamond$). A similar argument shows that $\Diamond_n$ cannot be defined from the standard modal operators. For an extended discussion of this fact refer to [5]. Furthermore, a similar argument shows that the modal operator M cannot be expressed with the graded modal operators. See Section 5 for a discussion.

As an example of the type of reasoning captured in our logic, consider the following variant of the well-known muddy children puzzle. Suppose that there are $n > 1$ children[†] who have been playing outside and $k > 1$ of them have mud on their forehead. After a while, the children's father comes outside and announces "A strict majority (strictly more than half) of you have mud on your forehead." The father then proceeds to ask the children to announce if they have dirt on their forehead. It is not too hard to see that the $(k - \lfloor \frac{n}{2} \rfloor)^{th}$ time[‡] the children are asked if they have mud on their

[†]Of course, we assume that the children are perfect reasoners, honest, and cannot feel the mud on their forehead.

[‡]Recall that $\lfloor \frac{n}{2} \rfloor$ is the integral part of $\frac{n}{2}$, i.e., the largest integer less than $\frac{n}{2}$.

forehead, the dirty children will correctly respond.

Given the intended interpretation of $W\alpha$, defining truth in a Kripke model is straightforward provided there are only finitely many accessible worlds. However, there are situations, such as in the canonical model, in which one cannot assume that the number of accessible worlds is finite. This leads us to the question of what is the majority of an infinite set? The standard definition, i.e. more than half, no longer makes sense. Should we consider the even numbers a weak majority of the natural numbers, and if so what about the set that contains all the even numbers plus the set $\{1, 3\}$? Mark Fey in [6] proposes a very interesting answer to this question. However, Fey's solution is not appropriate for our general framework and so we need another solution. In particular, Fey is concerned with majorities of countably infinite sets, whereas we would like a solution appropriate for any cardinality. Fey's approach is discussed in more detail in Section 4.1.2. We propose *majority spaces*, which generalizes the concept of an ultrafilter, as a solution to the problem of defining a majority of an infinite set.

This paper is organized as follows. The next section reviews graded modal logic. We then describe the language of majority logic and offer an axiomatization. After introducing majority spaces, we provide a Kripke style semantics and finally, in the last section, we prove completeness.

2. Graded Modal Logic

In this section, we provide a brief overview of graded modal logic. Graded modal logic was first introduced in [7, 8]. It was then studied in [3, 4, 5, 11, 16, 21, 20] in which issues of axiomatization, completeness, decidability and translations into predicate logic are discussed. We briefly discuss the language of graded modal logic and state some of the main results found in the literature. All results and proofs can be found in [5] and [3]. We first define the language of graded modal logic.

DEFINITION 2.1. Given a countable set of atomic propositions
$\mathbb{P} = \{p_0, p_1, \ldots\}$, define the language $\mathcal{L}_{GML}$ as the smallest set of formulas given by the following inductive definition:

$$\alpha := p \mid \neg\alpha \mid \alpha \vee \alpha \mid \Diamond_n\alpha$$

where $p \in \mathbb{P}$ and $n \in \mathbb{N}$.

For each $n \in \mathbb{N}$, we define $\Box_n\alpha := \neg\Diamond_n\neg\alpha$, and $\Diamond!_n\alpha := \Diamond_{n-1}\alpha \wedge \neg\Diamond_n\alpha$ $(n \neq 0)$ where $\Diamond!_0\alpha := \neg\Diamond_0\alpha$. So $\Diamond!_n\alpha$ will have the intended meaning that α is true in *exactly* n accessible worlds.

The following axiomatization was presented in [3].

G0 All tautologies in the language of **GML**

G1 $\Diamond_{n+1}\alpha \to \Diamond_n\alpha \quad (n \in \mathbb{N})$

G2 $\Box_0(\alpha \to \beta) \to (\Diamond_n\alpha \to \Diamond_n\beta) \quad (n \in \mathbb{N})$

G3 $\Diamond!_0(\alpha \wedge \beta) \to ((\Diamond!_{n_1}\alpha \wedge \Diamond!_{n_2}\beta) \to \Diamond!_{n_1+n_2}(\alpha \vee \beta)) \quad (n_1, n_2 \in \mathbb{N})$

Let **GML** be the smallest set of formulas of $\mathcal{L}_{\mathbf{GML}}$ that are instances of the above axiom schemes and that are closed under modus ponens (MP) and necessitation (N), i.e., from $\vdash \alpha$ infer $\vdash \Box_0\alpha$. We write $\vdash_{\mathbf{GML}} \alpha$ if $\alpha \in \mathbf{GML}$.

Formulas from $\mathcal{L}_{\mathbf{GML}}$ are interpreted over Kripke structures. Let $\mathcal{M} = \langle S, R, V\rangle$ be a Kripke model, where S is a set of worlds, R is a binary relation over S and $V : \mathbb{P} \to 2^S$ is a valuation function. The boolean connectives and propositional variables are evaluated as usual. We will only show how the formula $\Diamond_n\alpha$ is evaluated at a world $s \in S$:

$$\mathcal{M}, s \models \Diamond_n\alpha \text{ iff } |\{t : sRt \text{ and } \mathcal{M}, t \models \alpha\}| > n$$

We say α is valid in $\mathcal{M}$ iff $\forall s \in S$, $\mathcal{M}, s \models \alpha$, and write $\mathcal{M} \models \alpha$. We write $\models \alpha$ if α is valid in all models (based on some class of frames[§]) We also make use of the following notation throughout this paper: $R(s) = \{t \mid sRt\}$ and for any formula α, $R_\alpha(s) = \{t \mid sRt \text{ and } t \models \alpha\}$. So, the above definition can be rewritten as

$$\mathcal{M}, s \models \Diamond_n\alpha \text{ iff } |R_\alpha(s)| > n$$

Graded modal logic is shown to be sound and complete with respect to the class of all frames in [5]. Let $\mathfrak{F}$ be the class of all frames. It is easily verified that the axioms $G0 - G1$ are valid in any model based on $\mathfrak{F}$ and MP and N preserve validity. We state the completeness theorem below, but postpone discussion until section 6.

THEOREM 2.2 (Soundness and Completeness of **GML** [5]). *For any formula α of **GML**, $\models \alpha$ iff $\vdash_{\mathbf{GML}} \alpha$.*

In [3] the graded modal language is shown to be decidable by showing that it has the finite model property. Maarten de Rijke [4] arrives at the same

[§]Unless otherwise stated we will assume that we are working with models based on the class of all frames. Refer to [1] for more information on frames.

conclusion using an extended notion of bisimulation appropriate for a modal language with graded modalities. De Rijke also establishes invariance and definability results. Finally in [16], Tobbies shows that the decidability problem for **GML** is in $PSPACE$.

3. Majority Logic: Syntax

We extend the graded modal language ($\mathcal{L}_{\mathbf{GML}}$) with a new modal operator W. The formula $W\alpha$ is intended to mean that α is true in more than or equal to "half" of the number of accessible worlds.

DEFINITION 3.1. Given a countable set of atomic propositions $\mathbb{P} = \{p_0, p_1, \ldots\}$, define the language $\mathcal{L}_{\mathbf{MJL}}$ as the smallest set of formulas given by the following inductive definition:

$$\alpha := p \mid \neg\alpha \mid \alpha \vee \alpha \mid \Diamond_n \alpha \mid W\alpha$$

where $p \in \mathbb{P}$ and $n \in \mathbb{N}$.

Define $M\alpha := \neg W \neg \alpha$. So, $\mathcal{L}_{\mathbf{MJL}}$ takes the language $\mathcal{L}_{\mathbf{GML}}$ and closes under the operator W. Notice in particular that there are an infinite number of modal operators, one for each natural number plus the majority operator.

3.1. Axiomatization

We propose the following axiomatization. Since our language extends the language $\mathcal{L}_{\mathbf{GML}}$, we include the axiom schemes $G1, G2$ and $G3$. These axioms capture our intuitions about counting accessible worlds. But what axioms shall we adopt to reason about "majority"? The following discussion will motivate the proposed axiomatization which can be found at the end of the discussion.

Suppose a group of friends are trying to decide where to go for dinner. As is common in most social situations, the goal is to keep as many people happy as possible. Some of the friends prefer to eat Indian food and some prefer to eat vegetarian. To be more precise, some of the friends prefer to eat Indian food over not eating Indian food and similarly for vegetarian. Now, if more than half of the people prefer Indian and more than half prefer vegetarian for dinner, then there must be at least one person who wants an Indian vegetarian meal. This is easy to see if we consider a specific example. If there are 10 friends deciding on dinner and 6 people prefer Indian and 6 people prefer vegetarian, then obviously at least someone wants both Indian and vegetarian. This reasoning is captured by the following axiom scheme

$$M\alpha \wedge M\beta \rightarrow \Diamond(\alpha \wedge \beta)$$

Now, suppose that more than half of the friends prefer Indian for dinner. Also, suppose that every time the group eats Indian food, samosas are ordered. We can conclude that a majority of the friends want samosas. And so, we include the following axiom scheme

$$M\alpha \wedge \Box(\alpha \rightarrow \beta) \rightarrow M\beta$$

Suppose that you are put in charge of making dinner reservations for the group of 10 people. Say you are given the information that 5 people prefer Indian and 6 people prefer vegetarian. From this, as discussed above, right away you can conclude there is at least one person that prefers Indian and vegetarian. What if you are given the additional information that there are more than 3 people that do not prefer Indian and do not prefer vegetarian. The natural conclusion to draw is that more than 3 people prefer an Indian vegetarian meal. Otherwise, say you conclude that only 2 people prefer Indian and vegetarian. Let I denote the set of people that prefer Indian and V the set of people that prefer vegetarian. We know that since $|I| = 5$, $|V| = 6$ and $|I \cap V| = 2$, $|I \cap V^C| = 3$ and $|I^C \cap V| = 4$. these sets are disjoint, the total sum of people is 11 or more, and so it must be the case that more than 3 people like Indian and Italian. This line of reasoning is captured by the following axiom scheme

$$W\alpha \wedge W\beta \wedge \Diamond_n(\neg\alpha \wedge \neg\beta) \rightarrow \Diamond_n(\alpha \wedge \beta) \quad (n \in \mathbb{N})$$

The final situation is similar to the above situation except suppose that a majority of the people prefer Italian.

$$W\alpha \wedge M\beta \wedge \Diamond_n(\neg\alpha \wedge \neg\beta) \rightarrow \Diamond_{n+1}(\alpha \wedge \beta) \quad (n \in \mathbb{N})$$

The preceding discussion is summarized by the following list of axioms and rules.

AXIOM 1. *Classical propositional tautologies*

AXIOM 2. $\Diamond_{n+1}\alpha \rightarrow \Diamond_n\alpha \quad (n \in \mathbb{N})$

AXIOM 3. $\Box(\alpha \rightarrow \beta) \rightarrow (\Diamond_n\alpha \rightarrow \Diamond_n\beta) \quad (n \in \mathbb{N})$

AXIOM 4. $\Diamond!_0(\alpha \wedge \beta) \to ((\Diamond!_{n_1}\alpha \wedge \Diamond!_{n_2}\beta) \to \Diamond!_{n_1+n_2}(\alpha \vee \beta)) \quad (n_1, n_2 \in \mathbb{N})$

AXIOM 5. $M\alpha \wedge M\beta \to \Diamond(\alpha \wedge \beta)$

AXIOM 6. $M\alpha \wedge \Box(\alpha \to \beta) \to M\beta$

AXIOM 7. $W\alpha \wedge W\beta \wedge \Diamond_n(\neg\alpha \wedge \neg\beta) \to \Diamond_n(\alpha \wedge \beta) \quad (n \in \mathbb{N})$

AXIOM 8. $W\alpha \wedge M\beta \wedge \Diamond_n(\neg\alpha \wedge \neg\beta) \to \Diamond_{n+1}(\alpha \wedge \beta) \quad (n \in \mathbb{N})$

MP From α and $\alpha \to \beta$ derive β.

NEC From α derive $\Box\alpha$.

Let **MJL** be the smallest set of formulas that contain all instances of the above axiom schemes and are closed under the above rules. We write $\vdash_{\mathbf{MJL}} \alpha$ if α can be deduced from Axioms 1 - 8 using the rules MP and N (equivalently if $\alpha \in \mathbf{MJL}$). If it is clear from context, we may write $\vdash \alpha$ instead of $\vdash_{\mathbf{MJL}} \alpha$.

We now discuss some of the properties of the axioms proposed above. The first lemma given gives some consequences of the proposed axioms. Part (i) shows M and W are both monotone modal operators. Part (ii) is equivalent to saying that given any set X and any subset of X either it or its complement (or both) constitutes weak majority of X. (iii) are obvious properties of majority and weak majority sets. The proofs and a more detailed discussion can be found in [12].

LEMMA 3.2. *Suppose that α and β are arbitrary formulas from $\mathcal{L}_{\mathbf{MJL}}$. Then*

i. If $\vdash \alpha \to \beta$ then $\vdash M\alpha \to M\beta$ and $\vdash W\alpha \to W\beta$.

ii. $\vdash W\alpha \vee W\neg\alpha$

iii. $\vdash M\alpha \to W\alpha$ and $\vdash M\alpha \to \Diamond\alpha$

Using the language of graded modal logic, we can find a formula that expresses exactly how many worlds are accessible at any given state. For any $n \in \mathbb{N}$, the formula $\Diamond_n!\top$ will be true at some world w iff there are exactly n accessible worlds. The following lemma will be used in the completeness proof. The proof can be found in [12] and uses both Axiom 7 and Axiom 8.

LEMMA 3.3. *For all $n \in \mathbb{N}$, $\vdash \Diamond_n!\top \to (M\alpha \leftrightarrow \Diamond_{\lfloor n/2 \rfloor}\alpha)$.*

4. Majority Logic: Semantics

In this section we will present the semantics for $\mathcal{L}_{\mathbf{MJL}}$. The semantics will be an extension of the usual Kripke semantics. The formula $W\alpha$ will be true provided that the set of all accessible worlds in which α is true is a majority of the set of all accessible worlds. The definition makes sense only if there are *finitely* many accessible worlds. But what constitutes a majority of an infinite set? The following section offers a solution to this question.

Recall that if S is any set of states and R a binary relation on S, then $R(s) = \{t \mid sRt\}$ and for any formula α (from $\mathcal{L}_{\mathbf{MJL}}$ or $\mathcal{L}_{\mathbf{GML}}$), $R_\alpha(s) = \{t \mid sRt \text{ and } t \models \alpha\}$. This definition of course depends on the definition of truth in a model which is given below.

4.1. Majority Spaces

A very interesting situation arises when a Kripke model is not image finite, that is when $R(s)$ may be infinite for some state $s \in S$. While the semantics of a majority subset is very clear in the finite case, it is not clear what should constitute a majority when there are an infinite number of possibilities. We cannot for example stipulate that every infinite set is a (strict) majority. This would create the unsatisfactory situation where a set and its complement could be a strict majority.

Another natural choice would be to call a set $X \subseteq R(s)$ a majority if X^C is finite, i.e take the majority sets to be the co-finite sets. However, suppose that $R(s) = X_1 \cup X_2 \cup X_3$, where X_1, X_2, and X_3 are nonempty pairwise disjoint sets. Then one would expect that for some i and j where $i \neq j$, $X_i \cup X_j$ would be a majority. This is certainly true in the finite case, and so one would expect it to be true in the infinite case. However, it is easy to come up with an example where all of the X_i are infinite; and so, none of the $X_i \cup X_j$ would be a majority.

Instead of trying to define a majority set as some special subset of $R(s)$, we will let a model stipulate which sets are to be considered a majority. In other words, at each state in the model, we attach a collection of subsets of $R(s)$ which will be called the "majority" sets. Thus a set X in this collection will be considered a (weak) majority of $R(s)$ at state s. Obviously, we do not want to allow *any* collection of subsets, but only those collections that satisfying certain properties capturing our intuitions about majority.

Definition 4.1. Let W be any set. Call any set $\mathfrak{M} \subseteq 2^W$ a **majority system** if it satisfies the following properties.

M1. If $X \subseteq W$, then either $X \in \mathfrak{M}$ or $X^C \in \mathfrak{M}$.

M2. If $X \in \mathfrak{M}$, $Y \in \mathfrak{M}$ and $X \cap Y = \varnothing$, then $Y = X^C$.

M3. Suppose that $X \in \mathfrak{M}$ and $F \subseteq X$ is any finite set. If G is any set where $G \cap X = \varnothing$ and $|F| \leq |G|$, then $(X - F) \cup G \in \mathfrak{M}$.

The pair $\langle W, \mathfrak{M} \rangle$ will be called a **weak majority space**. Given a set W, a set $X \subseteq W$ will be called a **strict majority** (with respect to $\mathfrak{M}$) if $X \in \mathfrak{M}$ and $X^C \notin \mathfrak{M}$. X will be called a **weak majority** if $X \in \mathfrak{M}$ and $X^C \in \mathfrak{M}$. Call any set $X \in \mathfrak{M}$ a **majority set**. We first need to check that the above properties correspond to our intuitions about majority sets.

It is easy to see using M3 that majority spaces are closed under superset. We now show that many of the intuitions we have about majority sets on a finite space remain in a majority space. For example, we show that given any majority set X, if we add something new to X, then this newly formed set will be a strict majority. We also show that if a set W is infinite, then all majority sets must also be infinite.

Lemma 4.2. *If X is a weak majority and $F \neq \varnothing$ is a set such that $F \not\subseteq X$, then $X \cup F$ is a strict majority.*

Proof. Suppose that X is a weak majority and $F \neq \varnothing$ is any set such that $F \not\subseteq X$. Notice first that since $X \in \mathfrak{M}$ and $X \subseteq X \cup F$, then $X \cup F \in \mathfrak{M}$. We need only show that $(X \cup F)^C \notin \mathfrak{M}$. Suppose that $(X \cup F)^C \in \mathfrak{M}$. By property M2, since $X \in \mathfrak{M}$, $(X \cup F)^C \in \mathfrak{M}$ and $X \cap (X \cup F)^C = \varnothing$, we must have $(X \cup F)^C = X^C$ which implies $F \subseteq X$. But this contradicts the assumption that $F \not\subseteq X$. ■

Lemma 4.3. *Suppose that $\langle W, \mathfrak{M} \rangle$ is a majority space and that W is infinite. If $X \in \mathfrak{M}$ then X is infinite.*

Proof. Suppose that $\langle W, \mathfrak{M} \rangle$ is a majority space and W is infinite. Suppose that $X \subseteq W$ is finite and $X \in \mathfrak{M}$. Note that since X is finite, X^C is infinite. Take any finite set $G \subset X^C$, where $|X| \leq |G|$ (such a set must exist since W is infinite). Then by property M3, $(X - X) \cup G = G \in \mathfrak{M}$; and so, by property M2, $G = X^C$. But this is a contradiction since G is finite and X^C is infinite. ■

Corollary 1. *Suppose that $\langle W, \mathfrak{M} \rangle$ is a majority space and W is infinite. If X is a confinite set, then X is a strict majority.*

PROOF. Let $(W, \mathfrak{M})$ be a majority space (with W infinite) and $X \subseteq W$ a cofinite set. By property $M1$, to show that X is a strict majority, we need only show that $X^C \notin \mathfrak{M}$. But this follows directly from Lemma 4.3. ∎

Another way of stating this corollary is that if W is countable then for every $X \subseteq W$ if X is strictly larger[¶] than X^C, then X is a strict majority. So, one may wonder whether all majority spaces (where W is of arbitrary cardinality) have this property. To be more precise, we would like to have the following property:

> $(*)$ Let κ be an infinite cardinal. Suppose that $|W| = \kappa$ and $\langle W, \mathfrak{M} \rangle$ is a majority space. Then we have if $X \in \mathfrak{M}$, then $|X| = \kappa$.

The following example shows that if $\kappa = \omega_1$, then this property fails.

Example: Call a majority space $\langle W, \mathfrak{M} \rangle$ **strict** if every set $X \in \mathfrak{M}$ is a strict majority. Such spaces are studied in detail in [14]. Suppose that $\langle \mathbb{N}, \mathfrak{M}_{\mathbb{N}} \rangle$ is a strict majority space. We will construct a strict majority space on $\mathbb{R}$ that contains a countable set X. Define $\mathfrak{M}^0_{\mathbb{R}}$ as follows:

$$\mathfrak{M}^0_{\mathbb{R}} = \{X \mid X \cap \mathbb{N} \in \mathfrak{M}_{\mathbb{N}}\}$$

Now, it is easy to see that $\mathfrak{M}^0_{\mathbb{R}}$ has properties $M1$ and $M2$. First of all, notice that for any set $X \subseteq \mathbb{R}$, $X^C \cap \mathbb{N} = \mathbb{N} - (X \cap \mathbb{N})$. Let $X \subseteq \mathbb{R}$. Then either $X \cap \mathbb{N} \in \mathfrak{M}_{\mathbb{N}}$ or $X^C \cap \mathbb{N} = \mathbb{N} - (X \cap \mathbb{N}) \in \mathfrak{M}_{\mathbb{N}}$. Thus either $X \in \mathfrak{M}^0_{\mathbb{R}}$ or $X^C \in \mathfrak{M}^0_{\mathbb{R}}$. Furthermore, if $X \in \mathfrak{M}^0_{\mathbb{R}}$, then $X \cap \mathbb{N} \in \mathfrak{M}_{\mathbb{N}}$. Since $\langle \mathbb{N}, \mathfrak{M}_{\mathbb{N}} \rangle$ is a strict majority space, $X^C \cap \mathbb{N} = \mathbb{N} - (X \cap \mathbb{N}) \notin \mathfrak{M}_{\mathbb{N}}$. Hence, $X^C \cap \mathbb{N} \notin \mathfrak{M}^0_{\mathbb{R}}$. Thus, $M2$ is trivially satisfied. As for property $M3$, suppose that $X \in \mathfrak{M}^0_{\mathbb{R}}$ and $F \subseteq X$ is finite, $G \cap X = \emptyset$ and $|F| \le |G|$. Then if $|F \cap \mathbb{N}| > |G \cap \mathbb{N}|$, then $(X - F) \cup G$ might not be an element of $\mathfrak{M}^0_{\mathbb{R}}$. To rectify this situation, let $\mathfrak{M}_{\mathbb{R}} = \{A \mid A = (B - F) \cup G, B \in \mathfrak{M}^0_{\mathbb{R}}, F \subseteq B, F \text{ finite}, G \cap B = \emptyset, \text{ and } |F| \le |G|\}$. So, $\mathfrak{M}_{\mathbb{R}}$ is $\mathfrak{M}^0_{\mathbb{R}}$ closed under **finite perturbations**. A similar construction will be used later in Section 5. Thus, $\langle \mathbb{R}, \mathfrak{M}_{\mathbb{R}} \rangle$ is a strict majority space. Furthermore, for each $X \in \mathfrak{M}_{\mathbb{N}}$, $X \in \mathfrak{M}_{\mathbb{R}}$ by construction. Hence, there is a set X in $\mathfrak{M}_{\mathbb{R}}$ strictly smaller than its complement (the complement of a countable subset of $\mathbb{R}$ must be uncountable). ∎

[¶] Of course, in this context, "strictly larger" means that X is a cofinite set.

A similar construction can be used to show that there are non-principal ultrafilters over $\mathbb{R}$ that contain countable sets. The main roadblock to proving $(*)$ is that property $M3$ is not strong enough to outlaw examples like the one above. To that end, for each infinite cardinal κ, consider the following property:

$M3_\kappa$ Suppose $X \in \mathfrak{M}$ and $\lambda < \kappa$. Then if $F \subseteq X$ with $|F| = \lambda$ and G is any set with $G \cap X = \emptyset$ and $\lambda \leq |G|$, then $(X - F) \cup G \in \mathfrak{M}$.

So, property $M3$ is equivalent to $M3_{\aleph_0}$. Call a majority space $\langle W, \mathfrak{M}\rangle$ a κ-majority space if $\mathfrak{M}$ satisfies properties $M1$, $M2$ and $M3_\kappa$. Then the following lemma is a straightforward generalization of Lemma 4.3.

LEMMA 4.4. *Suppose that κ is an infinite cardinal, $|W| = \kappa$ and $\langle W, \mathfrak{M}\rangle$ is a κ-majority space. If $X \in \mathfrak{M}$, then $|X| = \kappa$.*

PROOF. Let κ be an infinite cardinal, $|W| = \kappa$ and $\langle W, \mathfrak{M}\rangle$ a κ-majority space. Further suppose that $X \in \mathfrak{M}$, but $|X| = \lambda$ for some $\lambda < \kappa$. Let $G \subseteq X^C$ be any set with $|G| = \lambda$ and $G \cap X = \emptyset$. Such a set must exist, since $\lambda < \kappa$, $|W| = \kappa$, $|X| = \lambda$ and hence $|W - X| = \kappa$. Then by property $M3_\kappa$, $(X - X) \cup G \in \mathfrak{M}$. Hence, by property $M2$ since $X \cap G = \emptyset$, $X = G^C$. But this is a contradiction, since $|G^C| = \kappa$, $|X| = \lambda$ and $\lambda < \kappa$.

■

For what follows, we only need $\aleph_0$-majority spaces, i.e., majority spaces. The last proposition demonstrates that our notion of an infinite majority is equivalent to the natural notion of a majority when we only have a finite number of elements. In other words, we will show that if W is a finite set, then the majority sets are the sets that have more than or equal to half of the elements.

PROPOSITION 4.5. *Suppose that W is a finite set and that $\mathfrak{M}' = \{M \subseteq W : |M| \geq |W|/2\}$, Then*

$$\langle W, \mathfrak{M}'\rangle \text{ is a majority space}$$

Furthermore, if $\langle W, \mathfrak{M}\rangle$ is any other majority space then $\mathfrak{M} = \mathfrak{M}'$.

PROOF. Suppose that W is a finite set and $\mathfrak{M}'$ is as defined above. We must first show that $\langle W, \mathfrak{M}'\rangle$ is a majority space. For any set, $X \subseteq W$, since $|X| + |X^C| = |W|$, either $|X| \geq |W|/2$ or $|X^C| \geq |W|/2$ and so either $X \in \mathfrak{M}'$ or $X^C \in \mathfrak{M}'$. Hence property M1 is satisfied. For property M2, suppose that $X, Y \in \mathfrak{M}'$, and $X \cap Y = \emptyset$. Since $|X| \geq |W|/2$ and

$|Y| \geq |W|/2$, $|X| + |Y| \geq |W|$. But since $X \cup Y \subseteq W$, $|X \cup Y| \leq |W|$ and so $|X \cup Y| = |W|$. Therefore, $X \cup Y = W$ (this follows since X and Y are assumed to be subsets of W). Since X and Y are disjoint and $X \cup Y = W$, then $Y = X^C$. Finally we need to show that property M3 is satisfied. Suppose that $X \in \mathfrak{M}'$. Then $|X| \geq |W|/2$. Suppose that $F \subseteq X$ and G is any finite set such that $|F| \leq |G|$ and $G \cap X = \emptyset$. Then

$$\begin{aligned} |(X - F) \cup G| &= |(X - F)| + |G| - |(X - F) \cap G| \\ &= |X - F| + |G| \\ &\geq |X - F| + |F| \\ &= |X \cup F| = |X| \geq |W|/2 \end{aligned}$$

so, $(X - F) \cup G \in \mathfrak{M}'$.

Let $\langle W, \mathfrak{M} \rangle$ be any majority space and let $X \in \mathfrak{M}$. We must now show that $|X| \geq |W|/2$. Suppose not, that is suppose that $|X| < |W|/2$. Therefore $|X^C| > |X|$. Let $Y \subseteq X^C$ and $|Y| = |X|$ (such a set must exist since $|X^C| > |X|$). Then by property M3, since $|X| \leq |Y|$ and $Y \cap X = \emptyset$, $(X - X) \cup Y = Y \in \mathfrak{M}$. But by property M2, $Y = X^C$. But this is a contradiction, since $|X| < |W|/2$ and $|Y| < |W|/2$. Hence, $|X| \geq |W|/2$. ■

4.1.1. Majority spaces and ultrafilters

Obviously majority spaces are closely related to ultrafilters. In fact in many papers on social choice theory on infinite populations, ultrafilters are used to capture the concept of "largeness", see [6, 15] for some examples. In this section, we study the connections between majority spaces and ultrafilters.

We first review some well-known definitions. Let W be a set. A non-empty collection $\mathcal{U} \subseteq 2^W$ is called an *filter* if $\mathcal{U}$ is closed under intersection and superset and does not contain the empty set. A filter is an *ultrafilter* if for all sets X either $X \in \mathcal{U}$ or $X^C \in \mathcal{U}$. Finally, $\mathcal{U}$ is *principal* if $\mathcal{U}$ contains a singleton, and $\mathcal{U}$ is non-principal if it is not principal (hence contains all cofinite sets). Given any infinite set, Zorn's lemma implies the existence of a non-principal ultrafilter.

Fix (an infinite) set W. At first non-principal ultrafilters seem to be a good candidate for the collection of majority subsets of W. Given any subset X, certainly either X or X^c should be considered a majority set; and majority sets are certainly closed under superset. However, it is easy to imagine a situation in which there are two majority sets whose intersection is not a majority. Thus ultrafilters should be thought of as the collection of "*vast* majorities", not majorities.

We first show that every non-principal ultrafilter is a majority system.

LEMMA 4.6. *Let W be an infinite set and $\mathcal{U}$ a non-principal ultrafilter over W. Then $\langle W,\mathcal{U}\rangle$ is a majority space.*

PROOF. Let W be an infinite set and $\mathcal{U}$ a non-principal ultrafilter over W. We need only show that $\mathcal{U}$ satisfies $M1, M2$ and $M3$. Obviously, $M1$ is satisfied. $M2$ is trivially satisfied since there are no sets $X, Y \in \mathcal{U}$ such that $X \cap Y = \emptyset$. We need only show that $M3$ is satisfied.

Let $Y = (X - F) \cup G$ where $X \in \mathcal{U}$, F is finite subset of X, $|F| \leq |G|$ and $X \cap G = \emptyset$. Since $\mathcal{U}$ is a non-principal ultrafilter then either $Y \in \mathcal{U}$ or $Y^C \in \mathcal{U}$. If $Y \in \mathcal{U}$ then we are done. Assume $Y^C \in \mathcal{U}$ then $Y^C \cap X \in \mathcal{U}$ and so $F \in \mathcal{U}$ which is a contradiction since F is finite. ■

Thus every ultrafilter is a majority system. We show below that the converse is not true. This is achieved by constructing an example of a majority system that is not an ultrafilter:

Example: Let X_1, X_2, X_3 be three disjoint infinite sets. Let $\mathfrak{U}_i$ be a non-principal ultrafilter over X_i for each $i = 1, 2, 3$. Now let $W = X_1 \cup X_2 \cup X_3$. Define

$$\mathfrak{M} = \{X | \exists i \neq j \text{ such that } X \cap X_i \in \mathfrak{U}_i \text{ and } X \cap X_j \in \mathfrak{U}_j\}$$

We claim that $(W, \mathfrak{M})$ is a majority space. We need only show that $\mathfrak{M}$ satisfies $M1 - M3$. First of all, notice that since the X_i are disjoint and each $\mathfrak{U}_i$ is an ultrafilter, for any set $Y \subseteq W$, either $Y \cap X_i \in \mathfrak{U}_i$ or $Y^C \cap X_i \in \mathfrak{U}_i$. Based on this observation, $M1$ follows easily:

Suppose that $X \subseteq W$. Assume $X \notin \mathfrak{M}$ then without loss of generality we can assume that $X \cap X_1 \notin \mathfrak{U}_1$ and $X \cap X_2 \notin \mathfrak{U}_2$. By the above observation, $X^C \cap X_1 \in \mathfrak{U}_1$ and $X^C \cap X_2 \in \mathfrak{U}_2$. So $X^C \in \mathfrak{M}$.

$M2$ is trivially true since the antecedent is always false. To see this, suppose that $X \in \mathfrak{M}, Y \in \mathfrak{M}$ and $X \cap Y = \emptyset$. Since the number of disjoint sets under consideration is odd, an easy application of the pigeon hole principle shows that there is an i such that $X \cap X_i \in \mathfrak{U}_i$ and $Y \cap X_i \in \mathfrak{U}_i$. So $(X \cap Y) \cap X_i \in \mathfrak{U}_i$ and thus $\emptyset \in \mathfrak{U}_i$ which is a contradiction.

Finally, $M3$ follows from the definition of ultrafilters and the following simple fact:

Fact Suppose $\mathfrak{U}$ is a non-principal ultrafilter and X any set. If $X \in \mathfrak{U}$ then for any finite set $F \subseteq X$, $X - F \in \mathfrak{U}$. Otherwise, $(X - F)^c \in \mathfrak{U}$ which implies $X^c \cup F \in \mathfrak{U}$. Therefore, $F = X \cap (X^c \cup F) \in \mathfrak{U}$, which is a contradiction since $\mathfrak{U}$ is a non-principal ultrafilter and so does not contain any finite sets.

Suppose that $X \in \mathfrak{M}$ and Let $Y = (X - F) \cup G$ where F is a finite subset of X, $X \cap G = \emptyset$ and $|F| \leq |G|$. Without loss of generality, we may assume that $X \cap X_1 \in \mathfrak{U}_1$ and $X \cap X_2 \in \mathfrak{U}_2$. Then, for each $i = 1, 2$, since F is a finite subset of X, $F \cap X_i$ is a finite subset of $X \cap X_i$. Therefore using the above fact, $X \cap X_i - (F \cap X_i) = (X - F) \cap X_i \in \mathfrak{U}_i$. Finally, since ultrafilters are closed under superset, for each $i = 1, 2$, $Y \cap X_i = ((X - F) \cup G) \cap X_i \in \mathfrak{U}_i$. Hence $Y \in \mathfrak{M}$.

Hence $(X, \mathfrak{M})$ is a majority space. Notice that $X_1 \cup X_2 \in \mathfrak{M}$ and $X_2 \cup X_3 \in \mathfrak{M}$ but their intersection $X_2 \notin \mathfrak{M}$. So $\mathfrak{M}$ is not an ultrafilter over X. It should be clear that this example can be generalized to any odd number of disjoint sets. ■

Thus we have shown that every ultrafilter is a majority system, but not every majority system is an ultrafilter. In fact, if we add the following axiom to the definition of a majority system then a majority system is equivalent to an ultrafilter.

M4 If $X, Y \in \mathfrak{M}$ then $X \cap Y \in \mathfrak{M}$.

Suppose that W is an infinite set, and $\mathfrak{M} \subseteq 2^W$ satisfies M1-M4. It is straightforward to check that $\mathfrak{M}$ is an ultrafilter. Notice that in the presence of M4, M2 is trivial. However, M2 and the fact that W has more than two elements is needed in order to show that $\emptyset \notin \mathfrak{M}$.

4.1.2. May's theorem

Of course, the question still remains as to whether our definition of a majority subset of an infinite set is "correct". The results and discussion of the previous two sections demonstrate that we have correctly generalized the familiar concept of a majority of a finite set to the infinite case. However, there is another direction we could go. In 1952, Kenneth May completely characterized simple majority rule for a finite set of individuals [9, 10]. Recently, Mark Fey generalizes this theorem to a countable set of individuals [6]. This section discusses how majority spaces relate to Fey's framework and May's celebrated result.

We first need some definitions. Fix an infinite set W. Suppose that there are two alternatives, x and y, under consideration. Elements of W are called voters. We assume that each voter has a linear preference over x and y, so for each $w \in W$, either w prefers x to y or y to x, but not both. Assume that a subset $X \subseteq W$, represents the set of all voters that prefer x to y. Thus X represents the outcome of a particular vote.

There are three possible outcomes to consider: 0 means that alternative y was chosen, $\frac{1}{2}$ means the vote was a tie, and 1 means that alternative x was chosen. An *aggregation function* is a function $f : 2^W \rightarrow \{0, \frac{1}{2}, 1\}$. Intuitively for a set $X \subseteq W$, $f(X)$ represents the social preference of the group W ($\frac{1}{2}$ is interpreted as a tie).

In [9, 10], May was concerned with which conditions on an aggregation function f force f to be equivalent to a simple majority decision. The first two conditions are the following. These are condition 3 and condition 4 from May's original paper ([9]) respectively. The terminology is also due to May.

DEFINITION 4.7.
1. An aggregation function f satisfies **neutrality** if, for all $X \subseteq W$, $f(X^C) = 1 - f(X)$
2. An aggregation function f satisfies **positive responsiveness** if, for all $X, Y \subseteq W$, $X \subsetneq Y$ and $f(X) \neq 0$ implies $f(Y) = 1$.

The last condition that May considers is **anonymity**[‖]. Anonymity essentially says that it is the number of votes that counts when determining the outcome, not *who* voted for what. When W is finite, this condition is straightforward to impose. Fix an arbitrary order on W, then each subset of W can be represented by a finite sequence of 1s and 0s. Then f satisfies anonymity if f is symmetric in this sequence of 1s and 0s. We will talk more about generalizing this condition below.

May showed that when W is finite, the conditions neutrality, positive responsiveness and anonymity[**] completely characterize the simple majority decision rule. Our goal in this section is to generalize this result to the infinite case using majority spaces. This is precisely Mark Fey's result in [6]. A formal comparison between Fey's framework and our framework would take us to far afield, and so will be reserved for a later paper.

Given any aggregation rule, define the collection $\mathfrak{M}_f$ of subsets of W as follows:

$$\mathfrak{M}_f = \{X \mid f(X) \geq \frac{1}{2}\}$$

[‖]May calls this condition equality.

[**]May has a fourth condition which essentially says that the aggregation function f is actually a function.

One result that we are after is that for any aggregation function satisfying neutrality, positive responsiveness and (an appropriate form) of anonymity, $(W, \mathfrak{M}_f)$ is a majority space. Conversely, given an majority space $(W, \mathfrak{M})$, we can construct an aggregation function $f_{\mathfrak{M}}$ as follows for each $X \subseteq W$,

$$f_{\mathfrak{M}}(X) = \begin{cases} 1 & \text{if } X^C \notin \mathfrak{M} \\ \frac{1}{2} & \text{if } X, X^C \in \mathfrak{M} \\ 0 & \text{if } X \notin \mathfrak{M} \end{cases}$$

Our desired result is that if $(W, \mathfrak{M})$ is a majority space, then $f_{\mathfrak{M}}$ satisfies May's three conditions. Of course, the appropriate generalization of May's theorem depends on an appropriate generalization of the definition of neutrality. Indeed, this constitutes the major portion of Fey's paper. We first deal with neutrality and positive responsiveness.

LEMMA 4.8. *If W is infinite and $(W, \mathfrak{M})$ is a majority space, then $f_{\mathfrak{M}}$ satisfies neutrality and positive responsiveness.*

PROOF. Neutrality is a consequence of $M1$ and the definition of $f_{\mathfrak{M}}$. Positive responsiveness is a consequence of Lemma 4.2. ∎

As for the converse,

LEMMA 4.9. *If f satisfies neutrality, then $\mathfrak{M}_f$ satisfies property $M1$.*

PROOF. Suppose that $X \notin \mathfrak{M}_f$ and $X^C \notin \mathfrak{M}_f$, then $f(X) = f(X^C) = 0$. By neutrality, $f(X^C) = 1 - f(X) = 1 - 0 = 1$. Contradiction. ∎

LEMMA 4.10. *If f satisfies positive responsiveness and neutrality, then $\mathfrak{M}_f$ satisfies property $M2$.*

PROOF. First note that $\emptyset \notin \mathfrak{M}_f$. Suppose $\emptyset \in \mathfrak{M}_f$. Then $f(\emptyset) \geq \frac{1}{2}$. By neutrality, $f(W) \leq \frac{1}{2}$. However, since $\emptyset \subsetneq W$ and $f(\emptyset) \geq \frac{1}{2}$, by positive responsiveness, $f(W) = 1$.

Suppose that X and Y are (nonempty) sets with $X \cap Y = \emptyset$ and $X, Y \in \mathfrak{M}_f$. We must show that $X = Y^C$. Suppose not. Then since $X \cap Y = \emptyset$, $X \subseteq Y^C$ and so $Y^C \not\subseteq X$. Hence, it must be the case that $X \subsetneq Y^C$. By positive responsiveness, since $f(X) \geq \frac{1}{2}$, $f(Y^C) = 1$. But this implies by neutrality that $f(Y) = 0$, contradicting the fact that $f(Y) \geq \frac{1}{2}$. ∎

We now turn to the subtle issue of generalizing May's anonymity condition. This condition says that an aggregation rule f should depend on the *size* of a set, not its contents. The problem, as shown by Cantor, is that every infinite subset of a (countably) infinite set has the same "size". And so, the intuition behind anonymity seems to imply that f should assign the same value to *every* infinite set.

Following [6], for the rest of this section only we assume that W is a countably infinite set. Fey's approach is to look to the set of permutations over W, i.e., the automorphism group over W. Recall that π a permutation if it is a 1-1 function from W to W. Then for any set $X \subseteq W$ and any given permutation π, define $\pi[X] = \{w \in W \mid \exists v \in X \text{ such that } \pi(v) = w\}$. Then we say that f satisfies **anonymity** provided $f(X) = f(\pi[X])$ for each permutation π. Clearly this condition is much too strong, since for any two infinite subsets of W we can find a permutation π such that $\pi[X] = Y$; and so if f satisfies anonymity, then every infinite subset must be assigned the same value. Fey considers two possible ways of restricting the set of permutation: finite permutations and bounded permutations. For a complete discussion of bounded permutations, the reader is referred to [6] and the references therein. We only discuss finite permutations.

DEFINITION 4.11. A permutation $\pi : W \to W$ is **finite** provided that there is a finite set $F \subseteq W$ such that $\pi(w) = w$ for each $w \in W - F$.

We say that an aggregation function satisfies **finite anonymity** provided $f(X) = f(\pi[X])$ for every finite permutation π. The last two observations will show that condition $M3$ corresponds to imposing finite anonymity, thus completing our generalization of May's theorem using majority spaces.

LEMMA 4.12. *If f satisfies finite anonymity and positive responsiveness, then $\mathfrak{M}_f$ satisfies $M3$.*

PROOF. Suppose that f satisfies finite anonymity, $X \in \mathfrak{M}_f$, F is a finite subset of X and G is any set such that $X \cap G = \emptyset$ and $|F| \leq |G|$. We must show that $(X-F) \cup G \in \mathfrak{M}_f$. Let $G' \subseteq G$ be any subset of G with $|F| = |G'|$ (such a set exists since $|F| \leq |G|$). Let π be the following permutation on W, $\pi(w) = w$ for each $w \in W - (F \cup G')$, and on $F \cup G'$ arrange it so that $\pi[F] = G'$. It is clear, that π is a finite permutation. Since f satisfies finite anonymity and $X \in \mathfrak{M}_f$, $f(\pi[X]) = f(X) \geq \frac{1}{2}$. If $(X-F) \cup G' = (X-F) \cup G$ then we are done, otherwise, $(X-F) \cup G' \subsetneq (X-F) \cup G$ and using positive responsiveness, $f((X-F) \cup G) = 1 \geq \frac{1}{2}$, and so $(X-F) \cup G \in \mathfrak{M}_f$. ■

LEMMA 4.13. *If $\mathfrak{M}$ satisfies $M3$, then $f_{\mathfrak{M}}$ satisfies finite anonymity.*

PROOF. Suppose that $\mathfrak{M}$ satisfies $M3$ and π is any finite permutation. We must show that $f_{\mathfrak{M}}(\pi[X]) = f_{\mathfrak{M}}(X)$ for each $X \subseteq W$. Given any set $X \subseteq W$, let $F_X = \{w \in X \mid \pi(w) \neq w\}$. Since π is a finite permutation, F_X is finite for every set X. It is easy to see that for any set $X \subseteq W$, $\pi[X] = (X - F_X) \cup \pi[F_X]$ and $X = (\pi[X] - \pi[F_X]) \cup F_X$. Hence, by property $M3$, $X \in \mathfrak{M}$ iff $\pi[X] \in \mathfrak{M}$. Using this fact, it is easy to check that $f_{\mathfrak{M}}$ satisfies finite anonymity. There are three cases to consider: $f_{\mathfrak{M}}(X) = 0$, $f_{\mathfrak{M}}(X) = \frac{1}{2}$ and $f_{\mathfrak{M}}(X) = 1$. The proofs are analogous, so only one case will be checked. The others are left to the reader. Suppose that $f_{\mathfrak{M}}(X) = 0$. We must show that $f_{\mathfrak{M}}(\pi[X]) = 0$. Suppose not, i.e., $f_{\mathfrak{M}}(\pi[X]) \geq \frac{1}{2}$. Then by construction of $f_{\mathfrak{M}}$, $\pi[X] \in \mathfrak{M}$, and so by the above discussion, $X \in \mathfrak{M}$. By the construction of $f_{\mathfrak{M}}$, this implies that $f_{\mathfrak{M}}(X) \geq \frac{1}{2}$, which is contradicts the assumption that $f(X) = 0$. ■

4.2. Majority Models

In this section we will extend the definition of a Kripke model in order to define truth of a majority logic formula.

DEFINITION 4.14. A **majority model** is a tuple $\mathcal{M} = \langle S, R, V, m \rangle$. Where S is any set of states, R is an accessibility relation and V is the valuation function $V : \mathbb{P} \to 2^S$, and $m : S \to 2^{2^S}$ is a **majority function** such that for each $s \in S$, $\langle R(s), m(s) \rangle$ is a majority space.

So, m assigns a majority space to each state. Let $s \in S$ be any state. We define the truth of a formula $\alpha \in \mathcal{L}_{\mathbf{MJL}}$ at state s in model $\mathcal{M}$ as follows:

1. $\mathcal{M}, s \models p$ iff $s \in V(p)$, where $p \in \mathbb{P}$
2. $\mathcal{M}, s \models \neg\alpha$ iff $\mathcal{M}, s \not\models \alpha$
3. $\mathcal{M}, s \models \alpha \vee \beta$ iff $\mathcal{M}, s \models \alpha$ or $\mathcal{M}, s \models \beta$
4. $\mathcal{M}, s \models \Diamond_n \alpha$ iff $|R_\alpha(s)| > n \qquad (n \in \mathbb{N})$
5. $\mathcal{M}, s \models W\alpha$ iff $R_\alpha(s) \in m(s)$

And so $\mathcal{M}, s \models M\alpha$ iff $R_{\neg\alpha}(s) \notin m(s)$. First notice that if $R(s)$ is finite for each $s \in S$, then by proposition 4.5, then $\mathcal{M}, s \models W\alpha$ iff $|R_\alpha(s)| \geq |R(s)|/2$. We now show that the axioms of majority logic are valid in all majority models.

THEOREM 4.15. ***MJL*** *is sound with respect to the class of all majority models.*

PROOF. Soundness was shown in [5] for axioms 1 - 4, MP, and Nec. Let $\mathcal{M} = \langle S, R, V, m \rangle$ be any majority model and $s \in S$. We will show Axiom 5 - 8 are true at state s. Since s is arbitrary, each axiom will be valid in $\mathcal{M}$; and hence, the axioms are sound. All of the proofs are straightforward and are left to the reader. As an example, we show the result holds for Axiom 7 and 8.

Axiom 7: Assume $s \models W\alpha \wedge W\beta \wedge \Diamond_n(\neg\alpha \wedge \neg\beta)$ so we have $R_\alpha(s) \in m(s), R_\beta(s) \in m(s)$ and $|R_{\neg\alpha\wedge\neg\beta}(s)| > n$. We must show $|R_{\alpha\wedge\beta}(s)| > n$. Assume $|R_{\alpha\wedge\beta}(s)| \leq n$. Let $X \subsetneq R_{\neg\alpha\wedge\neg\beta}(s)$ where $|X| = n$ (such a set exists since $|R_{\alpha\wedge\beta}(s)| > n$) and let $Y = (R_\beta(s) - R_{\alpha\wedge\beta}(s)) \cup X$. By $M3$, $Y \in m(s)$. It is easy to see that $Y \cap R_\alpha(s) = \emptyset$ and $Y \neq R_\alpha(s)^C = R_{\neg\alpha}(s)$. However, this contradicts $M2$ since $Y \in m(s)$ and $R_\alpha(s) \in m(s)$. So $|R_{\alpha\wedge\beta}(s)| > n$ and thus $s \models \Diamond_n(\alpha \wedge \beta)$.

Axiom 8: Assume $s \models W\alpha \wedge M\beta \wedge \Diamond_n(\neg\alpha \wedge \neg\beta)$ so we have $R_\alpha(s) \in m(s), R_{\neg\beta}(s) \notin m(s)$ and $R_{\neg\alpha\wedge\neg\beta}(s) > n$ we need to prove that $|R_{\alpha\wedge\beta}(s)| > n + 1$. Assume $|R_{\alpha\wedge\beta}(s)| \leq n + 1$. But $R_{\neg\beta}(s) = (R_\alpha(s) - R_{\alpha\wedge\beta}(s)) \cup R_{\neg\alpha\wedge\neg\beta}(s)$ and by M3 we get $R_{\neg\beta}(s) \in m(s)$ which is a contradiction.

■

Note that in a majority model, the majority function m is *any* function from states to majority sets. Of course, we may want to put some constraints on the majority functions we want to consider. For example, we can assume that for any states s and s', if $R(s) = R(s')$ then $m(s) = m(s')$. This and related issues will be left for a future paper.

5. Bisimulations for Majority Logic

In the introduction, we gave a quick argument that the graded modal operators cannot be expressed using the basic modal language. The point of the argument is that the graded modal language can distinguish between two bisimular models. We now present a similar argument which will show that the weak majority modal operator cannot be expressed using graded modal operators. In order to make this argument precise, we need a notion of bisimulation appropriate for graded modal logic. This was provided by Maarten de Rijke in [4]. De Rijke introduced g-bisimulations and used them to prove decidability, invariance and definability results about graded modal logic. In this section, we introduce m-bisimulations and show that m-bisimular models validate the same majority logic formulas.

First some notation. Suppose that $\mathcal{M}_1 = \langle S_1, R_1, m_1, V_1 \rangle$ and $\mathcal{M}_2 = \langle S_2, R_2, m_2, V_2 \rangle$ are majority models with $s \in S_1$ and $t \in S_2$. We say s and

t are g-**equivalent**, denoted $s \equiv_g t$, iff for every formula $\phi \in \mathcal{L}_{\mathbf{GML}}$, $s \models \phi$ iff $t \models \phi$, and say s and t are m-**equivalent**, denoted $s \equiv_m t$, iff for every formula $\phi \in \mathcal{L}_{\mathbf{MJL}}$, $s \models \phi$ iff $t \models \phi$. Obviously if s and t are m-equivalent, then they are g-equivalent.

DEFINITION 5.1. [4] Let $\mathcal{M}_1 = \langle S_1, R_1, m_1, V_1\rangle$ and $\mathcal{M}_2 = \langle S_2, R_2, m_2, V_2\rangle$ be two majority models. A g-**bisimulation** between $\mathcal{M}_1$ and $\mathcal{M}_2$ is an ω-length sequence of relations $Z = \langle Z_1, Z_2, \ldots\rangle$ satisfying the following requirements:

1. Z_1 is non-empty;
2. for all i, $Z_i \subseteq \mathcal{P}^{<\omega}(S_1) \times \mathcal{P}^{<\omega}(S_2)$;
3. if XZ_iY then $|X| = |Y| = i$;
4. if $\{s\}Z_1\{t\}$, then $V_1(s) = V_2(t)$;
5. if $\{s\}Z_1\{t\}$ and $X \subseteq R_1(s)$, where $|X| = i \geq 1$, then there exists $Y \in \mathcal{P}^{<\omega}(S_2)$ with $Y \subseteq R_2(t)$ and XZ_iY;
6. if $\{s\}Z_1\{t\}$ and $Y \subseteq R_2(t)$, where $|Y| = i \geq 1$, then there exists $X \in \mathcal{P}^{<\omega}(S_1)$ with $X \subseteq R_1(s)$ and XZ_iY;
7. if XZ_iY, then
   (a) for all $x \in X$ there exists $y \in Y$ with $\{x\}Z_1\{y\}$, and
   (b) for all $y \in Y$ there exists $x \in X$ with $\{x\}Z_1\{y\}$, and

Where $\mathcal{P}^{<\omega}(S)$ denotes the set of finite subsets of S. Let $Z : \mathcal{M}_1, s \underline{\leftrightarrow}_g \mathcal{M}_2, t$ denote that Z is a g-bisimulation with $\{x\}Z_1\{y\}$. The essential idea is that in order to preserve formulas of the form $\Diamond_n\phi$ the sets of successors of size n present in one model should be mirrored in the other model. We now can present an argument that g-bisimulations do not preserve all majority logic formulas.

Suppose that $S_1 = \{s\} \cup \mathbb{N}$ and $S_2 = \{t\} \cup \mathbb{N}$. Define the relation R_1 as sR_1n for each $n \in \mathbb{N}$ and the relation R_2 as tR_2n for each $n \in \mathbb{N}$. Thus $R_1(s) = \mathbb{N}$ and $R_2(t) = \mathbb{N}$. Let $m_1(s)$ be any non-principal ultrafilter over $\mathbb{N}$ containing the even numbers and $m_2(t)$ be any non-principal ultrafilter over $\mathbb{N}$ not containing the even numbers[††]. Since both $m_1(s)$ and $m_2(t)$

[††]Let E be the set of even numbers on O the set of odd numbers. Then it is easy to see that $\mathcal{F}_E = \{X \mid E \subseteq X\}$ and $\mathcal{F}_O = \{X \mid O \subseteq X\}$ are both filters; and so, can be extended to ultrafilters, say $\mathcal{U}_E$ and $\mathcal{U}_O$. Since $E \in \mathcal{F}_E$, $E \in \mathcal{U}_E$. Also, note that since $O \in \mathcal{U}_O$, $O^C = E \notin \mathcal{U}_O$.

are ultrafilters, by Lemma 4.6, both $\langle R_1(s), m_1(s)\rangle$ and $\langle R_2(t), m_2(t)\rangle$ are majority spaces. Suppose that the valuation of p be the set of even numbers in both models, i.e., $V_1(p) = V_2(p) = \{2n \mid n = 0, 1, \ldots\}$. Obviously, s and t are g-bisimular. However, since $V_1(p) \in m_1(s)$ and $V_2(p) \notin m_2(s)$, $\mathcal{M}_1, s \models Wp$ but $\mathcal{M}_2, t \not\models Wp$. Hence g-bisimularity does not imply m-equivalence. It is not hard to see, as shown in [4], that g-bisimularity implies g-equivalence:

THEOREM 2. *[4] Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two majority models and let Z be a g-bisimulation with $Z : \mathcal{M}_1, s \underline{\leftrightarrow}_g \mathcal{M}_2, t$, then $s \equiv_g t$.*

The proof is by induction on ϕ (see [4] for details). Below we will show how to extend g-bisimulations to a notion of bisimulation appropriate for majority logic.

DEFINITION 5.2. Let $\mathcal{M}_1 = \langle S_1, R_1, m_1, V_1\rangle$ and $\mathcal{M}_2 = \langle S_2, R_2, m_2, V_2\rangle$ be two majority models. An m-**bisimulation** is an ω-length tuple of relations $Z = \langle Z_0, Z_1, Z_2, \ldots\rangle$ with $Z_0 \subseteq (S_1 \times 2^{S_1}) \times (S_2 \times 2^{S_2})$ satisfying the following requirements:

1. $\langle Z_1, Z_2, \ldots\rangle$ is a g-bisimulation (Definition 5.1).
2. if $\{s\}Z_1\{t\}$ and there exists an $X \subseteq R_1(s)$ with $X \in m_1(s)$, then there is a Y such that $Y \subseteq R_2(t)$ and $(s, X)Z_0(t, Y)$;
3. if $\{s\}Z_1\{t\}$ and there exists an $Y \subseteq R_2(t)$ with $Y \in m_2(t)$, then there is an X such that $X \subseteq R_1(s)$ and $(s, X)Z_0(t, Y)$;
4. if $(s, X)Z_0(t, Y)$, then $X \in m_1(s)$ iff $Y \in m_2(s)$;
5. if $(s, X)Z_0(t, Y)$, then
   (a) for all $x \in X$ there exists $y \in Y$ with $\{x\}Z_1\{y\}$, and
   (b) for all $y \in Y$ there exists $x \in X$ with $\{x\}Z_1\{y\}$.

Let $Z : \mathcal{M}_1, s \underline{\leftrightarrow}_m \mathcal{M}_2, t$ denote that Z is an m-bisimulation with $\{s\}Z_1\{t\}$. The following theorem shows that our notion of m-bisimulation works as intended.

THEOREM 3. *Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two majority models and let Z be an m-bisimulation with $Z : \mathcal{M}_1, s \underline{\leftrightarrow}_n \mathcal{M}_2, t$, then $s \equiv_n t$.*

PROOF. The proof is by induction of ϕ. We must show for every majority logic formula ϕ, if $Z : \mathcal{M}_1, s \underline{\leftrightarrow}_m \mathcal{M}_2, t$, then $\mathcal{M}_1, s \models \phi$ iff $\mathcal{M}_2, t \models \phi$. The base case and boolean connectives are easy. The case when ϕ is of the form $\Diamond_n \psi$ follows from Theorem 2 and the fact that $\langle Z_1, \ldots \rangle$ is a g-bisimulation. We need only consider the case when ϕ is of the form $W\psi$. Suppose the result holds for ψ and $\mathcal{M}_1, s \models W\psi$. Then $R_{1_\psi}(s) \in m_1(s)$. Since $\{s\} Z_1 \{t\}$, by property 2 of Definition 5.2, there is a $Y \subseteq R_2(t)$ such that $(s, R_{1_\psi}(s)) Z_0 (t, Y)$. By property 4 of Definition 5.2, $Y \in m_2(t)$. Let $y \in Y$, then by the property 5 (b), there is an $x \in R_{1_\psi}(s)$ such that $\{x\} Z_1 \{y\}$. By the induction hypothesis, since $\mathcal{M}_1, x \models \psi$, $\mathcal{M}_2, y \models \psi$. Hence $y \in R_{2_\psi}(t)$. Therefore, $Y \subseteq R_{2_\psi}(t)$ and by property $M3$, since $Y \in m_2(t)$, $R_{2_\psi}(t) \in m_2(t)$. Hence $\mathcal{M}_2, t \models W\psi$. The other direction is similar. ■

By using m-bisimulations, the model theory of majority logic can be studied in a systematic manner. In particular, standard results from the model theory of modal logic such as invariance or definability results (see [1] for more information) can be studied. This and related topics will be left for further research.

6. Completeness

We first discuss completeness of graded modal logic. We will then adapt the proof to show completeness for **MJL**.

6.1. Completeness of Graded Modal Logic

Given any consistent set of formulas of majority logic, Γ, using Lindenbaum's Lemma, we can construct a maximally consistent superset of Γ. As usual, the states of our canonical model will be maximally consistent sets. In what follows, Γ will always be assumed to be a maximally consistent set of formulas.

When constructing a canonical model for a graded modal logic, it is necessary to control the number of worlds accessible from any given state. Given any state, i.e. maximally consistent set, Γ, our goal is to construct $R(\Gamma)$ such that

$$\Diamond_n \alpha \in \Gamma \text{ iff } |\{\Gamma' \in R(\Gamma) \mid \alpha \in \Gamma'\}| > n$$

Following [5], a satisfying family for each Γ, denoted by $SF(\Gamma)$ is constructed so that we may define $R(\Gamma) = SF(\Gamma)$ and then R will satisfy the above property. To this end we will present the following definitions and lemmas from [5]. Recall that ω is the first countable ordinal, and that $\omega+1 = \omega\cup\{\omega\}$. Let Φ be the set of all maximally consistent sets.

DEFINITION 6.1. The function $\mu : \Phi \times \Phi \to \omega + 1$ is defined as follows: for every $\Gamma_1, \Gamma_2 \in \Phi$

$$\begin{aligned} \mu(\Gamma_1, \Gamma_2) &= \omega \text{ if for any } \alpha \in \Gamma_2,\ \Diamond_n\alpha \in \Gamma_1 \text{ for all } n \in \mathbb{N} \\ \mu(\Gamma_1, \Gamma_2) &= \min\{n \in \mathbb{N} \ :\ \Diamond!_n\alpha \in \Gamma_1 \text{ and } \alpha \in \Gamma_2\} \text{ otherwise} \end{aligned}$$

That the function μ is well defined and for more properties of μ, the reader is referred to [3, 5]. The following lemma is an easy consequence of definition 6.1.

LEMMA 6.2. *Let $\Gamma_1, \Gamma_2 \in \Phi$. The following conditions are equivalent:*

1. $\mu(\Gamma_1, \Gamma_2) \neq 0$
2. *For any α, if $\alpha \in \Gamma_2$ then $\Diamond_0\alpha \in \Gamma_1$.*
3. *For any α, if $\Box_0\alpha \in \Gamma_1$ then $\alpha \in \Gamma_2$.*

The main idea is that μ will tell us how many accessible worlds are needed. Given two maximally consistent sets, Γ_1, Γ_2, $\mu(\Gamma_1, \Gamma_2)$ tells us the minimum number of copies of Γ_2 that are needed to be accessible from Γ_1. The following lemma shows that μ works as we expect.

LEMMA 6.3. *Let $\Gamma_1 \in \Phi$ and α be any formula*

1. *If $\Diamond_0\alpha \in \Gamma_1$ then there exists $\Gamma_2 \in \Phi$ such that $\alpha \in \Gamma_2$ and $\mu(\Gamma_1, \Gamma_2) \neq 0$.*
2. *If $\Diamond_n\alpha \in \Gamma_1$ for every $n \in \mathbb{N}$, then there exists $\Gamma_2 \in \Phi$ such that $\alpha \in \Gamma_2$ and $\mu(\Gamma_1, \Gamma_2) = \omega$.*

Refer to [5] for a proof. We are now ready to define the **satisfying family** of a maximally consistent set Γ_0.

DEFINITION 6.4. Let $\Gamma_0 \in \Phi$. The set

$$SF(\Gamma_0) = \bigcup\{\{\Gamma\} \times \mu(\Gamma_0, \Gamma) : \Gamma \in \Phi\}$$

will be called the satisfying family of Γ_0.

An element of $SF(\Gamma_0)$ is of the form $\langle\Gamma, n\rangle$ where $n < \mu(\Gamma_0, \Gamma)$, therefore we shall think of $SF(\Gamma_0)$ as made up of $\mu(\Gamma_0, \Gamma)$ ordered copies of Γ, for any $\Gamma \in \Phi$.
The following theorem is the main theorem from [5].

Theorem 6.5 ([5]). *For any α and any $n \in \mathbb{N}$,*

$$\Diamond_n \alpha \in \Gamma_0 \textit{ iff } |\{\Gamma \in SF(\Gamma_0) : \alpha \in \Gamma\}| > n$$

where to simplify notation, we write $\alpha \in \Gamma$ instead of $\alpha \in \langle \Gamma, n \rangle$.

6.2. Canonical Models for MJL

In this section we will define a canonical model for majority logic. We will now define the canonical model $\mathcal{M}^* = \langle S^*, R^*, V^*, m^* \rangle$ for **MJL** as follows: First of all, let

$$\mu(\Gamma) = \sup\{\mu(\Gamma', \Gamma) \mid \Gamma' \in \Phi\}$$

So $\mu(\Gamma)$ gives the maximum number of copies of Γ that will be needed in the canonical model. Define

$$S^* = \bigcup\{\{\Gamma\} \times \mu(\Gamma) \mid \Gamma \in \Phi\} \cup \{\langle \Gamma, 0 \rangle \mid \mu(\Gamma) = 0\}$$

So we may think of S^* as made up of $\mu(\Gamma)$ copies of Γ if $\mu(\Gamma) \neq 0$, and by one copy of Γ if $\mu(\Gamma) = 0$, for any maximally consistent set Γ.

For each $\langle \Gamma, i \rangle \in S^*$ define,

$$R^*(\langle \Gamma, i \rangle) = SF(\Gamma)$$

and for every proposition p and every $\langle \Gamma, i \rangle \in S^*$ we set:

$$V^*(p) = \{\langle \Gamma, i \rangle \mid p \in \Gamma\}$$

We need only define a majority function $m^* : S^* \to 2^{2^{S^*}}$. In what follows we will write Γ instead of $\langle \Gamma, i \rangle \in S^*$. This abuse of notation should not cause any confusion and so will be used to simplify the presentation. Let $R^*_\alpha(\Gamma) = SF_\alpha(\Gamma) = \{\Gamma' : \Gamma' \in SF(\Gamma) \text{ and } \alpha \in \Gamma'\}$. We are ready to define $m^*(\Gamma)$ so that $\langle R^*(\Gamma), m^*(\Gamma) \rangle$ is a majority space.

Given any maximally consistent set Γ, let $\mathfrak{M}_0(\Gamma) = \{SF_\alpha(\Gamma) \mid W\alpha \in \Gamma\}$. This is certainly a natural way to define a majority system given that we would like to prove a truth lemma. However, $\mathfrak{M}_0(\Gamma)$ will not in general be a majority system. The problem is with conditions $M3$. First note that every set in $\mathfrak{M}_0(\Gamma)$ is **definable**. A collection of maximally consistent sets X is **definable** provided there is a formula α (of majority logic) such that $X = \{\Gamma \mid \alpha \in \Gamma\}$. Take any set $X \in \mathfrak{M}_0(\Gamma)$ and let G be a nonempty collection of maximally consistent sets which is not definable and disjoint from X. Such a set surely exists since there are uncountably many maximally consistent sets, but only countably many formulas. Then by $M3$, $X \cup G$ should be an element of $\mathfrak{M}_0(\Gamma)$; however, there cannot be a single formula α such that $X \cup G = \{\Gamma \mid \alpha \in \Gamma\}$. Suppose there was such a formula, say α'. Since X is definable, by say α, then G would be definable by $\alpha' \wedge \neg\alpha$. Hence $X \cup G$ cannot be an element of $\mathfrak{M}_0(\Gamma)$. To ensure that the truth lemma goes through, we must have $\mathfrak{M}_0(\Gamma) \subseteq m^*(\Gamma)$.

As noted above, the sets in $\mathfrak{M}_0(\Gamma)$ are all definable. This fact will often be used in what follows, so some comments are in order. So, if $X \in \mathfrak{M}_0(\Gamma)$ then $X = SF_\alpha(\Gamma)$ for some formula α and $W\alpha \in \Gamma$. When we write X^C we mean $SF(\Gamma) - SF_\alpha(\Gamma)$; thus we have $X^C = SF_{\neg\alpha}(\Gamma)$. Similarly, if $X, Y \in \mathfrak{M}_0(\Gamma)$, with $X = SF_\alpha(\Gamma)$ and $Y = SF_\beta(\Gamma)$ and $W\alpha \in \Gamma$ and $W\beta \in \Gamma$ then $X \cup Y = SF_{\alpha\vee\beta}(\Gamma)$. Note that we do not necessarily have $X^C \in \mathfrak{M}_0(\Gamma)$ nor $X \cup Y \in \mathfrak{M}_0(\Gamma)$ if $X, Y \in \mathfrak{M}_0(\Gamma)$. Another point worth mentioning is the following: $SF_\phi(\Gamma) = SF_\psi(\Gamma)$ iff $\Box_0(\phi \leftrightarrow \psi) \in \Gamma$. Suppose that $\Box_0(\phi \leftrightarrow \psi) \in \Gamma$. By Lemma 6.2, for each $\Delta \in SF(\Gamma)$, $\phi \leftrightarrow \psi \in \Delta$. This implies that $SF_\phi(\Gamma) = SF_\psi(\Gamma)$. For the other direction, suppose that $SF_\phi(\Gamma) = SF_\psi(\Gamma)$ and $\Box_0(\phi \leftrightarrow \psi) \notin \Gamma$. Then since Γ is a maximally consistent set, $\Diamond_0\neg(\phi \leftrightarrow \psi) \in \Gamma$. Hence, by Lemma 6.3, there is a $\Delta \in SF(\Gamma)$ such that $\neg(\phi \leftrightarrow \psi) \in \Delta$. But this contradicts the fact that $SF_\phi(\Gamma) = SF_\psi(\Gamma)$.

We will now define the canonical majority function. It is easy to see that exactly one of the following cases must be true:

- $\Diamond!_n\top \in \Gamma$ for some $n \in \mathbb{N}$
- $\Diamond_n\top \in \Gamma\ \forall n \in \mathbb{N}$

If we are in Case 1, then $|SF(\Gamma)| = n$, and so we can define

$$m^*(\Gamma) = \{X \ : \ X \subseteq SF(\Gamma) \text{ and } |X| \geq \lceil |SF(\Gamma)|/2 \rceil\}$$

By Proposition 4.5, $\langle R(\Gamma), m^*(\Gamma)\rangle$ is a majority space. We need only check that $\mathfrak{M}_0(\Gamma) \subseteq m^*(\Gamma)$. This follows from Lemma 3.3 and Theorem 6.5.

Suppose that we are in case 2, that is for all $n \in \mathbb{N}$, $\Diamond_n\top \in \Gamma$. We first need some definitions.

Definition 6.6. Let X be any set, then $cof(X) = \{Y \subseteq X \mid Y^C \text{ is finite }\}$. So, $cof(X)$ is the set of co-finite subsets of X.

Definition 6.7. Let Y be any set and $X \subseteq 2^Y$. Then define

$$X^f = \{A \mid \exists B \in X \text{ such that } A = (B - F) \cup G \text{ where } F \text{ is finite, } |F| \leq |G| \text{ and } X \cap G = \varnothing\}$$

So, X^f is X **closed under finite perturbations**. It is easy to see that $X \subseteq X^f$ (take F and G both to be empty).

DEFINITION 6.8. Let Y be any set and $X \subseteq 2^Y$, then define

$$\overline{X} = \{A \ : \ A \notin X \text{ and } A^C \in X\}$$

Note that if $A \in X \cup \overline{X}$, then $A^C \in X \cup \overline{X}$; and hence if $A \notin (X \cup \overline{X})$, then $A^C \notin (X \cup \overline{X})$.

Let Γ be any maximally consistent set. We will now construct m^*:

1. Define $\mathfrak{M}_0(\Gamma) = \{SF_\alpha(\Gamma) \mid W\alpha \in \Gamma\}$
2. Define $\mathfrak{M}_1(\Gamma) = (\mathfrak{M}_0(\Gamma))^f$. That is take $\mathfrak{M}_0(\Gamma)$ and close off under finite perturbations.
3. Let $\mathcal{O} = SF(\Gamma) - (\mathfrak{M}_1(\Gamma) \cup \overline{\mathfrak{M}_1(\Gamma)})$. The set $\mathcal{O}$ contains the "other" sets. That is the sets X such that neither X nor X^C have made it into $\mathfrak{M}_1(\Gamma)$. In order to satisfy $M1$, we must pick one of X or X^C to be elements of $\mathfrak{M}_1$. These choices must be made in a way that is consistent with the properties $M1 - M3$. Let $\mathcal{U}$ be any non-principal ultrafilter over $SF(\Gamma)$. Define

$$m^*(\Gamma) = \mathfrak{M}_1(\Gamma) \cup (\mathcal{O} \cap \mathcal{U})$$

Before proving that $m^*(\Gamma)$ is in fact a majority system, we need some lemmas.

LEMMA 6.9. *Let Γ be any maximally consistent set. Suppose that $X, Y \in m^*(\Gamma)$ and $X \cap Y = \varnothing$. Then $X, Y \in \mathfrak{M}_1(\Gamma)$.*

PROOF. Let Γ be a maximally consistent set, and suppose that $X, Y \in m^*(\Gamma)$. Then by construction, there are four cases. If $X, Y \in \mathfrak{M}_1(\Gamma)$ then we are done. We need only show that the other three cases lead to a contradiction. Suppose that $X \in \mathcal{O} \cap \mathcal{U}$ and $Y \in \mathcal{O} \cap \mathcal{U}$. Then $X \cap Y \in \mathcal{U}$, which implies $\varnothing \in \mathcal{U}$. But this contradicts the fact the $\mathcal{U}$ is non-principal. Thus both X and Y cannot be elements of $\mathcal{O} \cap \mathcal{U}$. Suppose that $X \in (\mathcal{O} \cap \mathcal{U})$ and $Y \in \mathfrak{M}_1(\Gamma)$. Since $X \cap Y = \varnothing$, $Y \subseteq X^C$ so $X^C = Y \cup G$ for some set G. Therefore, $X^C \in \mathfrak{M}_1(\Gamma)$ which implies $X \in (\mathfrak{M}_1 \cup \overline{\mathfrak{M}_1})$. But this contradicts the assumption that $X \in \mathcal{O}$. Similarly we can show that $X \in \mathfrak{M}_1(\Gamma)$ and $Y \in \mathcal{O} \cap \mathcal{U}$ leads to a contradiction. ■

LEMMA 6.10. *If $X, Y \in \mathfrak{M}_0(\Gamma)$, then for every natural number n, if $|X^C \cap Y^C| > n$, then $|X \cap Y| > n$.*

PROOF. Suppose that $X, Y \in \mathfrak{M}_0(\Gamma)$, then there are formulas α and β such that $X = SF_\alpha(\Gamma)$, $Y = SF_\beta(\Gamma)$, $W\alpha \in \Gamma$ and $W\beta \in \Gamma$. Suppose $|X^C \cap Y^C| = |(SF_\alpha(\Gamma))^C \cap (SF_\beta(\Gamma))^C| = |SF_{\neg\alpha\wedge\neg\beta}(\Gamma)| > n$. Then $\Diamond_n(\neg\alpha \wedge \neg\beta) \in \Gamma$. Since $W\alpha \wedge W\beta \wedge \Diamond_n(\neg\alpha \wedge \neg\beta) \in \Gamma$, by axiom 7, $\Diamond_n(\alpha \wedge \beta) \in \Gamma$. Hence, $|X \cap Y| = |SF_\alpha(\Gamma) \cap SF_\beta(\Gamma)| > n$. ■

LEMMA 6.11. *If $X, Y \in \mathfrak{M}_1(\Gamma)$ and $X \cap Y = \emptyset$, then $X = Y^C$.*

PROOF. Suppose $X, Y \in \mathfrak{M}_1(\Gamma)$, then there are formulas α and β such that $W\alpha \in \Gamma$, $W\beta \in \Gamma$ and $X = SF_\alpha(\Gamma)$ and $Y = SF_\beta(\Gamma)$. We must show $X = Y^C$, that is $SF_\alpha(\Gamma) = SF_{\neg\beta}(\Gamma)$. This is equivalent to showing that $\Box_0(\alpha \leftrightarrow \neg\beta) \in \Gamma$. Since $X \cap Y = \emptyset$, $SF_{\alpha\wedge\beta}(\Gamma) = \emptyset$. Therefore, $\neg\Diamond_0(\alpha \wedge \beta) \in \Gamma$. By Axiom 7, $\neg\Diamond_0(\neg\alpha \wedge \neg\beta) \in \Gamma$. Hence, $\Box_0(\alpha \leftrightarrow \neg\beta) \in \Gamma$. ■

Finally, we show that $m^*(\Gamma)$ contains the cofinite subsets of $R^*(\Gamma)$.

LEMMA 6.12. *Let Γ be a maximally consistent set such that for all $n \geq 0$, $\Diamond_n\top \in \Gamma$. Then $cof(R^*(\Gamma)) \subseteq m^*(\Gamma)$.*

PROOF. Let $X \subseteq R^*(\Gamma)$ be cofinite. Then since $\mathcal{U}$ is a non-principal ultrafilter over $R^*(\Gamma)$, $X \in \mathcal{U}$. Hence if $X \in \mathcal{O}$, then $X \in m^*(\Gamma)$. Suppose that $X \notin \mathcal{O}$, then either $X \in \mathfrak{M}_1(\Gamma)$ or $X^C \in \mathfrak{M}_1(\Gamma)$. If $X \in \mathfrak{M}_1(\Gamma)$, then we are done. Thus if we can show that $X^C \notin \mathfrak{M}_1(\Gamma)$ then we are done. Suppose that $X^C \in \mathfrak{M}_1(\Gamma)$. Then there is a $Z \in \mathfrak{M}_0(\Gamma)$ such that $X^C = (Z - F) \cup G$, where $G \cap Z = \emptyset$, $F \subseteq Z$ and $|F| \leq |G|$. Since X^C is finite, then Z is finite. Thus there is some $k \geq 0$ such that $|Z| = k$. Hence there is a formula α such that $W\alpha \in \Gamma$ and $|Z| = |\{\Delta \mid \Delta \in SF(\Gamma) \text{ and } \alpha \in \Delta\}| = k$. Hence by Theorem 6.5, $\Diamond!_k\alpha \in \Gamma$.

We first show that in **MJL** (actually, this can be done in **GML**),

$$(*) \qquad \vdash \Diamond!_k\alpha \wedge \Diamond_{2k}\top \to \neg\Diamond!_k\neg\alpha$$

As an instance of $G3$, $\vdash \Diamond!_0(\alpha \wedge \neg\alpha) \to (\Diamond!_k\alpha \wedge \Diamond!_k\neg\alpha) \to \Diamond!_{2k}(\alpha \vee \neg\alpha))$. Since $\vdash \Diamond!_0\bot$ (this formula is equivalent to $\Box_0\top$), using propositional reasoning $\vdash (\Diamond!_k\alpha \wedge \Diamond!_k\neg\alpha) \to \Diamond!_{2k}\top$. Since $\vdash \Diamond!_{2k}\top \to \neg\Diamond_{2k}\top$, using propositional reasoning $\vdash (\Diamond!_k\alpha \wedge \Diamond!_k\neg\alpha) \to \neg\Diamond_{2k}\top$ and hence using propositional reasoning $\vdash (\Diamond!_k\alpha \wedge \Diamond_{2k}\top) \to \neg\Diamond!_k\neg\alpha$.

Therefore, $(\Diamond!_k\alpha \wedge \Diamond_{2k}\top) \to \neg\Diamond!_k\neg\alpha \in \Gamma$. Since $\Diamond!_k\alpha \in \Gamma$ and by assumption $\Diamond_{2k}\top \in \Gamma$, we have $\neg\Diamond!_k\neg\alpha \in \Gamma$. Hence either $\neg\Diamond_{k-1}\neg\alpha \in \Gamma$ or $\Diamond_k\neg\alpha \in \Gamma$. Since $W\alpha \wedge \Diamond_k\neg\alpha \to \Diamond_k\alpha$ is an instance of Axiom 7, $W\alpha \wedge \Diamond_k\neg\alpha \to \Diamond_k\alpha \in \Gamma$. Suppose $\Diamond_k\neg\alpha \in \Gamma$. Then since $W\alpha \in \Gamma$, we have $\Diamond_k\alpha \in \Gamma$. However, since $\Diamond!_k\alpha \in \Gamma$, we have $\neg\Diamond_k\alpha \in \Gamma$, which contradicts the fact that Γ is a maximally consistent set. Thus we are done if we can show $\neg\Diamond_{k-1}\neg\alpha \notin \Gamma$.

Notice that we can generalize the proof of $(*)$ to show that for each $j \geq 0$,

$$\vdash \Diamond!_k\alpha \wedge \Diamond_{k+j}\top \rightarrow \neg\Diamond!_j\neg\alpha$$

Since for each $j \geq 0$, $\Diamond_{k+j}\top \in \Gamma$, we have for each $j \geq 0$, $\neg\Diamond!_j\neg\alpha \in \Gamma$. Now if $j = 0$, $\neg\Diamond!_j\neg\alpha$ is by definition $\Diamond_0\neg\alpha$. Hence $\Diamond_0\neg\alpha \in \Gamma$. We show by induction on j that for $j > 1$, $\Diamond_j\neg\alpha \in \Gamma$. The base case is $j = 1$. Since $\neg\Diamond!_j\neg\alpha \in \Gamma$. Either $\neg\Diamond_0\neg\alpha \in \Gamma$ or $\Diamond_1\neg\alpha \in \Gamma$. Since $\Diamond_0\neg\alpha \in \Gamma$, we have $\Diamond_1\neg\alpha \in \Gamma$. Suppose the statement is true for $j = l$. We must show $\Diamond_{l+1}\neg\alpha \in \Gamma$. The argument is analogous to the base case, since $\neg\Diamond!_{l+1}\neg\alpha \in \Gamma$, either $\neg\Diamond_l\neg\alpha \in \Gamma$ or $\Diamond_{l+1}\neg\alpha \in \Gamma$. By the induction hypothesis, $\Diamond_l\neg\alpha \in \Gamma$, hence $\Diamond_{l+1}\neg\alpha \in \Gamma$. Therefore, we have $\Diamond_{k-1}\neg\alpha \in \Gamma$. Hence $\neg\Diamond_{k-1}\neg\alpha \notin \Gamma$.

■

Also note that the above proof can also be used to show that if for all $n \geq 0$, $\Diamond_n\top \in \Gamma$ and $X \in m^*(\Gamma)$, then X is infinite. Otherwise, since X is finite and $\mathcal{U}$ is a non-principal ultrafilter, $X \notin \mathcal{U}$ and therefore, $X \in \mathfrak{M}_1(\Gamma)$. But the above proof shows that we can derive a contradiction under the assumption that X is finite. Using the above lemmas, we can now show that the above construction of m^* gives us a majority space. That is, we show that $\langle R^*(\Gamma), m^*(\Gamma)\rangle$ is a majority space.

Lemma 6.13. *Given any maximally consistent set* Γ, $\langle R^*(\Gamma), m^*(\Gamma)\rangle$ *is a majority space.*

Proof. Let Γ be any maximally consistent set. Then we are either in case 1 or case 2 (as stated above). If we are in case 1, then $\langle R^*(\Gamma), m^*(\Gamma)\rangle$ is a majority space by Proposition 4.5. Thus we may assume that we are in case 2, and so $R^*(\Gamma)$ is infinite. We must show $m^*(\Gamma)$ satisfies properties $M1, M2$, and $M3$.

(M1) Let $X \subseteq R^*(\Gamma)$. We must show that either $X \in m^*(\Gamma)$ or $X^C \in m^*(\Gamma)$. By construction, either $X \in \mathcal{O}$ or $X \in \mathfrak{M}_1(\Gamma) \cup \overline{\mathfrak{M}_1(\Gamma)}$. Suppose that $X \in \mathcal{O}$. Then $X^C \in \mathcal{O}$. By the definition of an ultrafilter, either $X \in \mathcal{U}$ or $X^C \in \mathcal{U}$. Say $X \in \mathcal{U}$. Then by construction, $X \in m^*(\Gamma)$. The result is similar if $X^C \in \mathcal{U}$. Suppose that $X \in \mathfrak{M}_1(\Gamma) \cup \overline{\mathfrak{M}_1(\Gamma)}$. Then either $X \in \mathfrak{M}_1(\Gamma)$ or $X \in \overline{\mathfrak{M}_1(\Gamma)}$. If $X \in \mathfrak{M}_1(\Gamma)$ then we are done. If $X \in \overline{\mathfrak{M}_1(\Gamma)}$, then $X^C \in m^*(\Gamma)$. In either case either $X \in m^*(\Gamma)$ or $X^C \in m^*(\Gamma)$.

(M2) Suppose that $X, Y \in m^*(\Gamma)$ and $X \cap Y = \emptyset$ we must show $X = Y^C$. Since $X, Y \in m^*(\Gamma)$ and $X \cap Y = \emptyset$, by Lemma 6.9, both X and Y are elements of $\mathfrak{M}_1(\Gamma)$.

Suppose that $X^C \neq Y$. By construction there are sets $Z_1, Z_2 \in \mathfrak{M}_0(\Gamma)$, finite sets $F_1 \subseteq Z_1, F_2 \subseteq Z_2$ and sets G_1, G_2 such that $|F_1| \leq |G_1|$, $|F_2| \leq |G_2|$, $G_1 \cap Z_1 = \emptyset$, $G_2 \cap Z_2 = \emptyset$ and

$$X = (Z_1 - F_1) \cup G_1 \qquad \text{and} \qquad Y = (Z_2 - F_2) \cup G_2$$

Let $a = |Z_1 \cap Z_2| = |F_1 \cap F_2|$ (since $X \cap Y = \emptyset$), $b = |F_1 \cap G_2|$ and $c = |F_2 \cap G_1|$.

We first show that $Z_1 \cap Z_2 = \emptyset$. Suppose not, that is suppose that $a \neq 0$. By construction of X and Y and the fact that $X \cap Y = \emptyset$, we have $(G_1 - F_2) \cup (G_2 - F_1) \subseteq Z_1^C \cap Z_2^C$ (and $(G_1 - F_2) \cap (G_2 - F_1) = \emptyset$). Hence $|(G_1 - F_2) \cup (G_2 - F_1)| \leq |Z_1^C \cap Z_2^C|$. Therefore, $|Z_1^C \cap Z_2^C| \geq |(G_1 - F_2) \cup (G_2 - F_1)| \geq |G_1 - F_2| + |G_2 - F_1| \geq |G_1| - b + |G_2| - c \geq |F_1| + |F_2| - (b + c) \geq a + b + a + c - (b + c) = 2 \times a > a$ (since $a \neq 0$). This implies that $|Z_1^C \cap Z_2^C| > |Z_1 \cap Z_2|$ and so (since $|Z_1 \cap Z_2|$ is finite) there is a natural number n such that $|Z_1 \cap Z_2| \leq n$ and $|Z_1^C \cap Z_2^C| > n$, which contradicts Lemma 6.10.

Therefore $Z_1 \cap Z_2 = \emptyset$, and so by Lemma 6.11 $Z_1 = Z_2^C$. Then $X \cup Y = (Z_1 - F_1) \cup G_1 \cup (Z_2 - F_2) \cup G_2 = ((Z_1 \cup Z_2) - (F_1 \cup F_2)) \cup G_1 \cup G_2 = (W - F') \cup G'$ where $F' = F_1 \cup F_2$ is a finite subset of W and $G' = G_1 \cup G_2$. So we get $F_1 \cup F_2 = G_1 \cup G_2$ and so $X = Y^C$, as desired.

(M3) Let $X \in m^*(\Gamma)$ and $Y = (X - F) \cup G$ where F is finite subset of X, $|F| \leq |G|$ and $X \cap G = \emptyset$. First of all, note that if X is cofinite, then Y is cofinite, so by Lemma 6.12, $Y \in m^*(\Gamma)$. Thus, we need only consider the case when both X and X^C are infinite in what follows[‡‡]. We need to prove that $Y \in m^*(\Gamma)$.

- If $X \in \mathfrak{M}_1(\Gamma)$, then there is a $Z \in \mathfrak{M}_0(\Gamma)$ such that $X = (Z - F') \cup G'$, where F' is a finite subset of Z, $|F'| \leq |G'|$. We will prove that $Y = (Z - F'') \cup G''$ where F'' is a finite subset of Y and $|F''| \leq |G''|$ and $G'' \cap Z = \emptyset$. We have, $Y = (Z - ((F' - G) \cup (F - G'))) \cup (G - F') \cup (G' - F)$ which implies that $Y \in \mathfrak{M}_1(\Gamma)$.

[‡‡]The fact that X cannot be finite is discussed after the proof of Lemma 6.12.

– If $X \in (\mathcal{O} \cap \mathcal{U})$ then $X \in \mathcal{O}$ and $X \in \mathcal{U}$. Since $\mathcal{U}$ is a non-principal ultrafilter, $Y \in \mathcal{U}$. Otherwise, $Y^C \in \mathcal{U}$ and so $X \cap Y^C = F \in \mathcal{U}$ which contradicts the fact that $\mathcal{U}$ is non-principal. If $Y \in \mathcal{O}$ then we are done. Assume $Y \notin \mathcal{O}$ then $Y \in (\mathfrak{M}_1(\Gamma) \cup \overline{\mathfrak{M}_1(\Gamma)})$. Hence either $Y \in \mathfrak{M}_1(\Gamma)$ or $Y^C \in \mathfrak{M}_1(\Gamma)$. If $Y \in \mathfrak{M}_1(\Gamma)$, then we are done since this implies $Y \in m^*(\Gamma)$. Suppose $Y^C \in \mathfrak{M}_1(\Gamma)$. Then if we can show that $X^C \in \mathfrak{M}_1(\Gamma)$, then we are done since this contradicts the assumption that $X \in \mathcal{O}$. Thus all that remains is to show $X^C \in \mathfrak{M}_1(\Gamma)$.

  Since $Y^C \in \mathfrak{M}_1(\Gamma)$, there is some $Z \in \mathfrak{M}_0(\Gamma)$ such that $Y^C = (Z - F') \cup G'$ where $F' \subseteq Z$, $G' \cap Z = \emptyset$ and $|F'| \leq |G'|$. Recall that $Y = (X - F) \cup G$ where $F \subseteq X$, $G \cap X = \emptyset$ and $|F| \leq |G|$. We claim that $X^C = (Z - (F' \cup (Z \cap F))) \cup ((G' \cap F^C) \cup (G \cap F'^C))$. First we must show that the right hand side of the above equality satisifies the requisite properties. Clearly, $(F' \cup (Z \cap F)) \subseteq Z$. Furthermore, $G' \cap Z = \emptyset$, so $(G' \cap F^C) \cap Z = \emptyset$. We also have $(G \cap F'^C) \cap Z = \emptyset$. Otherwise, there is a $\Delta \in Z$ such that $\Delta \in G \cap F'^C$. Since $\Delta \in Z$ and $\Delta \notin F'$, we have $\Delta \in Z - F' \subseteq Y^C$. But since $\Delta \in G$, we have $\Delta \in Y$, a contradiction. Finally, note that without loss of generality we can assume that both G and G' are infinite. Essentially, this follows because we are assuming that both X and X^C are infinite (and hence so are Y and Y^C). Then it is not hard to show that $|F' \cup (Z \cap F)| \leq |(G' \cap F^C) \cup (G \cap F'^C)|$ (if G and G' are infinite then so is $(G' \cap F^C) \cup (G \cap F'^C)$ while $F' \cup (Z \cap F)$ is finite).

  We must show 1. $X^C \subseteq (Z - (F' \cup (Z \cap F))) \cup ((G' \cap F^C) \cup (G \cap F'^C))$ and 2. $(Z - (F' \cup (Z \cap F))) \cup ((G' \cap F^C) \cup (G \cap F'^C)) \subseteq X^C$. Note that since $Y = (X - F) \cup G$, $Y^C = (X^C \cup F) \cap G^C$.

  1. Suppose that $\Delta \in X^C$. Either $\Delta \in G$ or $\Delta \in G^C$. Suppose $\Delta \in G$. Since $\Delta \in X^C$, $\Delta \in X^C \cup F \subseteq Y^C$. Therefore, either $\Delta \in Z - F'$ or $\Delta \in G'$. If $\Delta \in Z - F'$, then obviously $\Delta \notin F'$. Since $G' \cap Z = \emptyset$ and $F' \subseteq Z$, if $\Delta \in G'$, then $\Delta \notin F'$. In either case, $\Delta \in G \cap F'^C$. Hence $\Delta \in (Z - (F' \cup (Z \cap F))) \cup ((G' \cap F^C) \cup (G \cap F'^C))$. Suppose $\Delta \in G^C$. Then since $\Delta \in X^C \subseteq (X^C \cup F)$, $\Delta \in (X^C \cup F) \cap G^C = Y^C$. Hence either $\Delta \in Z - F'$ or $\Delta \in G'$. If $\Delta \in Z - F'$, then since $\Delta \notin F$ (as $F \subseteq X$ and $\Delta \in X^C$), $\Delta \in Z - (F' \cup (Z \cap F))$.

Suppose $\Delta \in G'$. Since $\Delta \in X^C$, we have $\Delta \in F^C$. Thus $\Delta \in G' \cap F^C \subseteq (Z-(F' \cup (Z \cap F))) \cup ((G' \cap F^C) \cup (G \cap F'^C))$. In either case, $\Delta \in (Z-(F' \cup (Z \cap F))) \cup ((G' \cap F^C) \cup (G \cap F'^C))$.

2. Suppose that $\Delta \in (Z-(F' \cup (Z \cap F))) \cup ((G' \cap F^C) \cup (G \cap F'^C))$. There are three cases to consider:
   (a) Suppose $\Delta \in G \cap F'^C$. Then $\Delta \in G$ and since $X \cap G = \emptyset$, we have $\Delta \in X^C$.
   (b) Suppose $\Delta \in G' \cap F^C$. Then since $\Delta \in G'$, $\Delta \in Y^C \subseteq (X^C \cup F) \cap G^C$. Therefore, $\Delta \in X^C \cup F$. Hence, either $\Delta \in X^C$ or $\Delta \in F$. However, we assumed $\Delta \in F^C$. Therefore, $\Delta \in X^C$.
   (c) Suppose $\Delta \in (Z - (F' \cup (Z \cap F)))$. Then since $(Z - (F' \cup (Z \cap F))) \subseteq (Z - F')$, $\Delta \in Z - F' \subseteq Y^C$. Therefore, $\Delta \in Y^C \subseteq X^C \cup F$. Hence either $\Delta \in X^C$ or $\Delta \in F$. But since $\Delta \in (Z - (F' \cup (Z \cap F)))$, $\Delta \notin F$. Hence $\Delta \in X^C$.

   In any of the above cases, $\Delta \in X^C$ and so, $(Z - (F' \cup (Z \cap F)) \cup (G' \cap F^C) \cup (G \cap F'^C)) \subseteq X^C$.

■

THEOREM 6.14. *For any maximally consistent set* Γ *and any formula* α *of* ***MJL****,*

$$R^*_\alpha(\Gamma) \in m^*(\Gamma) \textit{ iff } W\alpha \in \Gamma.$$

PROOF. Let Γ be a maximally consistent set and α any formula of **MJL**.

($\Leftarrow$) Suppose $W\alpha \in \Gamma$ then $R^*_\alpha(\Gamma) \in \mathfrak{M}_0(\Gamma) \subseteq m^*(\Gamma)$. Thus, $R^*_\alpha(\Gamma) \in m^*(\Gamma)$.

($\Rightarrow$) Suppose $R^*_\alpha(\Gamma) \in m^*(\Gamma)$. Since Γ is maximally consistent, by Lemma 3.2 part 2, $W\alpha \vee W\neg\alpha \in \Gamma$. Therefore, we need only show that $W\neg\alpha \in \Gamma$ and $W\alpha \notin \Gamma$ leads to a contradiction. Suppose that $W\neg\alpha \in \Gamma$ and $W\alpha \notin \Gamma$. Since $W\neg\alpha \in \Gamma$, by construction $R^*_{\neg\alpha}(\Gamma) \in \mathfrak{M}_0(\Gamma)$. Hence $R^*_\alpha(\Gamma) \notin \mathcal{O}$. Therefore, $R^*_\alpha(\Gamma) \in \mathfrak{M}_1(\Gamma)$.

Since $R^*_\alpha(\Gamma) \in \mathfrak{M}_1(\Gamma)$, there is a set $Z \in \mathfrak{M}_0(\Gamma)$, a finite set $F \subseteq R^*_\alpha(\Gamma)$ and a set $G \subseteq R^*(\Gamma)$ such that $|F| \leq |G|$ and

$$R^*_\alpha(\Gamma) = (Z - F) \cup G \qquad (*)$$

Thus, there is some formula β such that $W\beta \in \Gamma$ and $Z = R^*_\beta(\Gamma)$. Suppose that $|F| = k$ for some integer k. Then $R^*_{\neg\alpha\wedge\beta}(\Gamma) = (R^*_\alpha(\Gamma))^C \cap$

$R^*_\beta(\Gamma) = F$. Therefore, $|R^*_{\neg\alpha\wedge\beta}(\Gamma)| = k$. By Lemma 6.5, $\Diamond!_k(\neg\alpha\wedge\beta) \in \Gamma$. Hence (1) $\Diamond_{k-1}(\neg\alpha \wedge \beta) \in \Gamma$ and (2) $\neg\Diamond_k(\neg\alpha \wedge \beta) \in \Gamma$. Since $G \cap Z = \varnothing$, we have $G \subseteq R^*_{\alpha\wedge\neg\beta}(\Gamma)$. Since $|F| \leq |G|$, we have $k \leq |F| \leq |G| \leq |R^*_{\alpha\wedge\neg\beta}(\Gamma)$. Again by Lemma 6.5, $\Diamond_{k-1}(\alpha\wedge\neg\beta) \in \Gamma$. Since $W\alpha \notin \Gamma$ and Γ is maximally consistent, $M\neg\alpha \in \Gamma$. Thus by axiom 8, since $W\beta \in \Gamma$, $M\neg\alpha \in \Gamma$ and $\Diamond_{k-1}(\alpha \wedge \neg\beta) \in \Gamma$, $\Diamond_k(\neg\alpha \wedge \beta) \in \Gamma$. But this contradicts (2). Therefore it cannot be the case that $W\neg\alpha \in \Gamma$ and $W\alpha \notin \Gamma$. Hence $W\alpha \in \Gamma$.

■

Given the previous lemmas, completeness is straightforward. We give some of the details.

LEMMA 6.15 (Truth Lemma). *For any formula α and any $\Gamma \in S^*$ we have*

$$\mathcal{M}^*, \Gamma \models \alpha \text{ iff } \alpha \in \Gamma.$$

PROOF. The proof is by induction on the complexity of α. The proof is trivial for the base case and boolean connectives. We will show the result for the modal formulas.

1. Suppose $\alpha = \Diamond_n\beta$. $\mathcal{M}^*, \Gamma \models \Diamond_n\beta$ iff by definition of truth in a model $|R^*_\beta(\Gamma)| > n$ iff by Theorem 6.5 $\Diamond_n\beta \in \Gamma$.

2. Suppose $\alpha = W\beta$. $\mathcal{M}^*, \Gamma \models W\beta$ iff (by the induction hypothesis) $R^*_\beta(\Gamma) \in m^*(\Gamma)$ iff (by Theorem 6.14) $W\beta \in \Gamma$.

■

Given the truth lemma for **GML** and **MJL**, the completeness theorem follows using a standard argument.

THEOREM 6.16 (Canonical Model Theorem for **MJL**). *Let $\mathcal{M}^*$ be the canonical model described above, then for any formula α of majority logic, $\vdash_{\boldsymbol{MJL}} \alpha$ iff α is valid in $\mathcal{M}^*$.*

7. Conclusion and Future Work

We have extended graded modal logic with an operator W that can express the concept of weak majority. In order to interpret W in a Kripke structure, we defined a majority space. A majority space extends the well-defined concept of a majority of a finite set to an infinite set. A axiom system was presented and shown to be both sound and complete.

Along the way, we looked at how to define the majority of an infinite set. Instead of trying to find a naturally occurring definition, we define a majority space which gives a lot of room in the definition of a majority subset of an infinite set. So if asked if the even numbers ($\mathbb{E}$) are a strict majority or a weak majority of the natural numbers ($\mathbb{N}$), we would answer that it depends on what is being modeled. On the one hand, it seems clear that $\mathbb{E}$ is a weak majority of $\mathbb{N}$. However, consider the following sequence of sets: $\{0, 2, 1\}, \{0, 2, 4, 1, 3\}, \{0, 2, 4, 6, 1, 3, 5\}, \ldots$. The first set has a strict majority of even numbers, and since each new set adds only one even number and one odd number, every element of this sequence has a strict majority of even numbers. The limit of this sequence is $\mathbb{N}$; and so if we think of $\mathbb{N}$ as being "constructed" by this sequence of sets, one would expect that $\mathbb{E}$ is a *strict* majority. Essentially, this example shows that the well-ordering of the natural numbers plays an important role in determining which sets are considered majorities.

The main technical question is the decidability of **MJL**. Since it was shown in [3] that graded modal logic has the finite model property, we expect that **MJL** will share this property.

Another important area for further research is to relate the work in this paper to work on *generalized quantifiers*. Given a first-order model $\mathbb{M}$ with domain D, a generalized quantifier (of type 1) is interpreted as a collection of subsets of D. That is if Q is a generalized quantifier, then $Q^{\mathbb{M}}$ (the interpretation of Q in $\mathbb{M}$) is a subset of 2^D. Then the interpretation of the formula $Qx\phi(x)$ is as follows

$$\mathbb{M} \models Qx\phi(x) \text{ iff } \{x \mid \mathbb{M} \models \phi(x)\} \in Q^{\mathbb{M}}$$

Thus, $\forall^{\mathbb{M}} = \{D\}$ and $\exists^{\mathbb{M}} = 2^D - \emptyset$. A common example of a generalized quantifier is Most, where the interpretation of $\mathsf{Most}\ x\phi(x)$ is "more than half of the objects in D satisfy ϕ." Typically, when studying such an operator, it is assumed that the domains are finite. See [18] for a discussion of generalized quantifiers over infinite domains and [17] for a discussion of generalized quantifiers over finite domains. Majority spaces can be used to provide a interpretation of $\mathsf{Most}\ x\phi(x)$ in models with infinite domains. An interesting avenue of future research is to study known results about the Most quantifier from the point of view of majority logic. A number of authors have noted such a parallel between modal logic and generalized quantifiers. For example, see [22, 19] for discussions on the connections between generalized quantifiers and modal logic.

We also point out that we cannot express the statement “among the worlds in which α is true, β is a majority ” in our language. Such statements are often used when reasoning about candidates in an election. For example, among the Democratic registered voters, Kerry has the majority of their votes. We would like to extend the language of majority logic with an operator that can express such statements. A step in this direction would be to introduce a binary modality $\leq$, in which the intended meaning of $\alpha \leq \beta$ is α is true in “less” states than β.

Finally, we point to some possible applications of our logic. Although, the primary interest of this paper is technical, we feel that our framework can be used to reason about social software (see [13] for more information) such as voting systems [2].

8. Acknowledgments

Both authors would like to thank Rohit Parikh for many useful comments and suggesting that this paper be written. We also thank the Knowledge, Games and Beliefs group at CUNY for allowing us to present this work at various stages of its development. The authors would like to thank Joel Hamkins, Yde Venema, Noson Yanofsky and a anonymous referee for very useful comments and suggestions. Finally, we thank an anonymous referee for a number of excellent suggestions which greatly improved the readibaility and content of this paper.

References

[1] Blackburn, P., de Rijke, M., and Venema, Y. *Modal Logic.* Cambridge University, 2001.

[2] Brams, S., and Fishburn, P. voting procedures. *Handbook of social science and welfare 1* (Aug 2002).

[3] Caro, F. D. Graded modalities II (canonical models). *Studia Logica 47* (1988), 1–10.

[4] de Rijke, M. A note on graded modal logic. *Studia Logica 64* (2000), 271–283.

[5] Fattorosi-Barnaba, M., and Cerrato., C. Graded modalities I. *Studia Logica 47* (1988), 99–110.

[6] Fey, M. May's theorem with an infinite population. Research report, University of Rochester, 2002.

[7] Fine, K. In so many possible worlds. *Notre Dame Journal of formal logic 13*, 4 (1972), 516–520.

[8] Goble, L. Grades of modality. *Logique et Analyse 13* (1970), 323–334.

[9] May, K. A set of independent necessary and sufficient conditions for simple majority decision. *Econometrica 20* (October 1952), 680 – 684.

[10] May, K. A note on the complete independence of the conditions for simple majority decision. *Econometrica 21* (1953), 172 – 173.

[11] Ohlbach, H. J., Schmidt, R. A., and Hustadt, U. Proof theory and modal logic. In *Studies in Applied Logic*, H. Wansing, Ed. Kluwer, 1995, ch. Translating graded modalities into predicate logic, pp. 245–285.

[12] Pacuit, E., and Salame, S. Majority logic. In *Principles of Knowledge Representation and Reasoning: Proceedings of the Ninth International Conference (KR2004), Whistler, Canada, June 2-5, 2004* (2004), D. Dubois, C. A. Welty, and M.-A. Williams, Eds., AAAI Press, pp. 598–605.

[13] Parikh, R. Social software. *Synthese 132* (Sept 2002), 187–211.

[14] Salame, S. *Majoriy Spaces and its Connections with Ultrafilters*. PhD thesis, City University of New York, Graduate Center, 2005.

[15] Taylor, A. D. Infinite voter populations: ultrafilter generalizations of theorems of arrow and Gibbard-Satterthwaite. unpublished, 2004.

[16] Tobies, S. PSPACE reasoning for graded modal logics. *Journal of Logic and Computation 11*, 1 (2001), 85–106.

[17] Väänänen, J. Generalized quantifiers. *Bulletin of the European Association for Theoretical Computer Science 1754* (1999).

[18] Väänänen, J. Barwise: Abstract model theory and generalized quantifiers. *Bulletin of Symbolic Logic 10*, 1 (March 2004).

[19] van Benthem, J. Guards, bounds, and generalized semantics. *Journal of Logic, Language and Information 14*, 3 (June 2005), 263 – 279.

[20] van der Hoek, M., and de Rijke, M. Counting objects. *Journal of Logic and Computation 5*, 3 (1995), 325 – 345.

[21] van der Hoek, W. On the semantics of graded modalities. *Journal of Applied Non-Classical Logics 2*, 1 (1992), 81 – 123.

[22] van der Hoek, W., and de Rijke, M. Generalized quantifiers and modal logic. *Journal of Logic, Language, and Information 2* (1993), 19 – 50.

Eric Pacuit
Institute of Logic, Language and Computation
University of Amsterdam
Plantage Muidergracht 24
1018 TV AMSTERDAM
epacuit@science.uva.nl

Samer Salame
Department of Computer Science
Graduate Center of CUNY
365 5th Avenue
New York City, NY 10016
ssalame@gc.cuny.edu