Changing Types

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The central thesis of the epistemic program in game theory (Brandenburger, 2007) is that the basic mathematical models of a game situation should include an explicit parameter describing the players' *informational attitudes*[1]. Games are played in specific *informational contexts*, in which players have specific knowledge and beliefs about each other[2]. Many different formal models have been used to represent such informational contexts of a game (see Bonanno and Battigalli (1999); van der Hoek and Pauly (2006); van Benthem et al. (2011), and references therein, for a discussion). In this paper, we are not only interested in structures that describe the informational context of a game, but how these structures can *change* in response to the players' observations, communicatory acts or other dynamic operations of information change (cf. van Benthem, 2010).

We focus our attention on the players' *hard information* about the game (which we refer to as *knowledge* following standard terminology in the game theory and epistemic logic literature) and its dynamics. Two main structures have been used to model the players' knowledge about a game situation. Both structures start with a set S of states of nature (these are intended to represent possible outcomes of a game situations). It is typically assumed that elements of S can be described by some logical language (for example, propositional logic). The first type of models are the so-called *Aumann-* or *Kripke-structures* (Aumann, 1999; Fagin et al., 1995). These structures describe the players' knowledge in terms of an *epistemic indistinguishability* relations over a finite set of states W. The second type of model we are interested in are the knowledge structures of (Fagin et al., 1991, 1999), which are non-probabilistic variants of *Harsanyi type spaces* (Harsanyi, 1967)[3]. States in a knowledge structure are called **types** and explicitly describe the players' infinite hierarchy of knowledge (the 0th-level knowledge about the ground fact, 1st-level knowledge about the knowledge of the other players, 2nd-level knowledge about the other players 1st-level knowledge, and so on). The precise relationship between these two structures was clarified by Fagin et al. (1991, 1999).

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[1]This is, of course, something of a truism regarding games of *incomplete* or *imperfect* information. But, the thesis is intended to apply to *all* game situations.

[2]This is nicely explained by Adam Brandenburger and Amanda Friedenberg (2010, pg. 801): "In any particular structure, certain beliefs, beliefs about beliefs, ..., will be present and others won't be. So, there is an important implicit assumption behind the choice of a structure. This is that it is "transparent" to the players that the beliefs in the [type] structure — and only those beliefs — are possible....The idea is that there is a "context" to the strategic situation (eg., history, conventions, etc.) and this "context" causes the players to rule out certain beliefs."

[3]See Siniscalchi (2008) for a modern introduction to type spaces and Myerson (2004) for a discussion of Harsanyi's classic paper.

Recent work in *dynamic epistemic logic* focuses on operations that *change* a Kripke model. These operations are intended to represent different types of *epistemic actions*, such as (uncertain) observations or different types of communicatory events (such as public announcements). The logical theory of these dynamic operations is very well-developed as explained in two recent books (van Ditmarsch et al., 2007; van Benthem, 2010). The main goal of this paper is to explore the ramifications of this theory of information change for knowledge structures (and, eventually, for Harsanyi type spaces). For instance, given any two types (based on the same state of nature) in a knowledge structure, can we always find operations that can transform one type into the other? In the remainder of this short abstract, we sketch some of the key ideas that are important for our analysis.

A Primer on Dynamic Epistemic Logic. We assume the reader is familiar with the basics of (dynamic) epistemic logic (see Fagin et al., 1995; van Benthem, 2010, for introductions), and so, we only give the key definitions here. Let I be the set of players and At a finite set of atomic propositions[4]. The basic epistemic language, denoted $\mathcal{L}_{EL}$, is the smallest set of formulas generated by the following grammar

$$\varphi \ ::= \ p \mid \neg\varphi \mid \varphi \wedge \varphi \mid K_i\varphi$$

where $p \in \mathsf{At}$ and $i \in I$. The other boolean connectives ($\vee, \rightarrow$) are defined as usual, and the intended interpretation of $K_i\phi$ is "agent i knows that ϕ is true". Formulas of $\mathcal{L}_{EL}$ are interpreted at states in a Kripke structures. A **Kripke structure** is a tuple $\langle W, \{R_i\})i \in I, V\rangle$ where W is a (finite) set of states, $R_i \subseteq W \times W$ is an equivalence relation[5], and $V : \mathsf{At} \rightarrow \wp(W)$ is a valuation function assigning to each atomic proposition a set of states. Formulas of $\mathcal{L}_{EL}$ are interpreted at states in a Kripke model, we only remind the reader of the definition of truth of the knowledge modality:

$$\mathcal{M}, w \models K_i\phi \text{ iff } \text{ for all } v \text{ with } wR_iv : \mathcal{M}, v \models \varphi$$

In this paper, we concentrate on (and generalize) the product update with event models (also called action models): An **event model** is a tuple $\langle E, \{S_i\}_{i\in I}, \mathsf{pre}\rangle$ where E is a (finite) set of atomic events, $S_i \subseteq E \times E$ is a relation on E and $\mathsf{pre} : E \rightarrow \mathcal{L}_{EL}$ assigns to each primitive event a formula that serves as a **precondition** for that event. In this paper (to simplify the presentation), we restrict attention to event models where the relations S_i are equivalence relations. A Kripke model $\mathcal{M}$ incorporates the information described in an event model $\mathcal{E}$ via a product update. The **product update** of a Kripke model $\mathcal{M} = \langle W, \{R_i\}_{i\in I}, V\rangle$ and an event model $\mathcal{E} = \langle E, \{S_i\}_{i\in I}, \mathsf{pre}\rangle$ is a Kripke model $\mathcal{M} \oplus \mathcal{E} = \langle W', \{R'_i\}_{i\in I}, V'\rangle$ defined as follows:

- $W' = \{(w, e) \in W \times V : \mathcal{M}, w \models \mathsf{pre}(e)$
- $(w, e)R'_i(w', e')$ iff wR_iw' and eS_ie'
- $(w, e) \in V'(p)$ iff $w \in V(p)$

Knowledge Structures. A state in a knowledge structure is a vector of functions $\langle f_0, f_1, f_2 \ldots\rangle$ called a κ-world where κ (a possibly infinite ordinal) denotes the length of

[4] Atomic propositions are intended to represent properties of states of nature.

[5] In this paper, we restrict attention structures where the epsitemic relations are equivalence relations. These are known in the literature as **S5**-structures or Aumann structures.

the vector. Let $\mathcal{F}_\kappa(S)$ be the set of all κ-worlds over state space S. The precise definition of a κ-world is inductive:

- A 1-world $\langle f_0 \rangle$ contains the state of nature that obtains ($f_0 \in S$).
- For all $\alpha > 0$ the component f_α is a function from the set of agents I to the power set of the set of α-worlds over S ($f_\alpha : I \to \wp(\mathcal{F}_\alpha(S))$ satisfying some conditions. The intended interpretation is that $f_\alpha(i)$ is the set of all α-worlds player i considers possible. The conditions for the f_α are:
  - *correctness* $\langle f_0 \dots f_{n-1} \rangle \in f_n(i)$, i.e. every agent must consider the actual state of the world possible
  - *introspection* If $\langle g_0 \dots g_{n-1} \rangle \in f_n(i)$ then $g_1(i) = f_1(i) \dots g_{n-1}(i) = f_{n-1}(i)$, i.e. the player knows about his lower-order beliefs insofar as he cannot consider states possible where his assessments are different from what they actually are
  - *extendability* If $0 < \alpha < \beta$ then $g \in f_\alpha(i)$ iff there is some $h \in f_\beta(i)$ such that g is an initial segment of h i.e. all α worlds are approximations of a 'complete' belief and thus have to be extendable to an arbitrary length.

Fagin et al. (1999) prove a number of sophisticated results giving the precise relationship between Kripke structures and knowledge structures. We can related Kripke structures to knowledge structures as follows: Suppose that $\mathsf{At} = \{p_1, \dots, p_n\}$ and the state of nature is assumed is defined as follows $S = \wp(\mathsf{At})$ (S consists of all possible truth assignments to the propositional letters). For every agent i the indistinguishability of two states for i defines a partition of $\mathcal{F}_\alpha(S)$, denote the corresponding equivalence relation with $\sim_i$. Thus $f \sim_i g$ iff $f_\beta(i) = g_\beta(i)$ for all $\beta < \alpha$. Then $\langle \mathcal{F}_\alpha(S), \{\sim_i\}_{i \in N}, V \rangle$ is an (infinite) epistemic model where we set $f \in V(p)$ provided $p \in f_0$.

Now, for every epistemic model $\mathcal{M} = \langle W, \{R_i\}_{i \in N}, V \rangle$ there is natural type map r_α from W to $\mathcal{F}_\alpha(S)$ defined by: $r(w)_0 = \{p \mid w \in V(p)\}$ and $r(w)_{\alpha+1}(i) = \{r(v)_\alpha(i) \mid wR_iv\}$. The map r_α is always a total simulation[6] from $\mathcal{M}$ to the epistemic model $\langle \mathcal{F}_\alpha(S), \{\sim_i\}_{i \in N}, V \rangle$. Thus, studying the epistemic model $\langle \mathcal{F}_\alpha(S), \{\sim_i\}_{i \in N}, V \rangle$ can provide some information about all epistemic models[7]. Note that, in general, r_α is not a bisimulation onto its image, but the following holds:

- if $\mathcal{M}$ is finite and $\alpha \geq \omega$, then r_α is a bisimulation onto its image. Moreover, $r_\omega(\mathcal{M})$ is the bisimulation minimal model of $\mathcal{M}$
- For any epistemic models $\mathcal{M}$ and $\mathcal{M}'$ with w a state in $\mathcal{M}$ and w' a state in $\mathcal{M}'$:

$$r_\omega(w) = r_\omega(w') \Leftrightarrow \text{for all } \varphi \in \mathcal{L}_{EL}, \quad \mathcal{M}, w \models \varphi \text{ iff } \mathcal{M}', w' \models \varphi$$

[6] A simulation satisfies one-half of the definition of a bisimulation: Let $\mathcal{M} = \langle W, \{R_i\}_{i \in I}, V \rangle$ and $\mathcal{M}' = \langle W', \{R'_i\}_{i \in I}, V' \rangle$ be Kripke structures. A nonempty relation $Z \subseteq W \times W'$ is a simulation from $\mathcal{M}$ to $\mathcal{M}'$ provided wZw' iff w and w' satisfy the same atomic formulas and for each $i \in I$ and $v \in W$, if wR_iv then there exists a $v' \in W'$ such that $w'R'_iv'$ and vZv'. The relation Z is total provided for each $w \in W$ there is a $w' \in W'$ such that wZw'.

[7] The epistemic model $\langle \mathcal{F}_\alpha(S), \{\sim_i\}_{i \in N}, V \rangle$ is not a terminal object in the category of epistemic models with bisimulations as morphisms: For $\alpha \neq \beta$ the maps $r^\beta_\alpha : \langle \mathcal{F}_\beta(S), \dots \rangle \to \langle \mathcal{F}_\alpha(S), \dots \rangle$ and $r^\alpha_\beta : \langle \mathcal{F}_\alpha(S), \dots \rangle \to \langle \mathcal{F}_\beta(S), \dots \rangle$ are not inverse to each other. Heifetz and Samet (1998) have shown in that there is no universal object in this category.

Our Contribution. Our aim is to model the reasoning processes that take place in type spaces by examining natural *transitions* between types. For this initial study, we focus on product update. First, we define a sequence of products $\times_n$ between *Kripke structures*[8]. With these products, we give an simple proof of the following theorem of van Ditmarsch and French (2009).

Theorem 1 (van Ditmarsch and French, 2009) *Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be Kripke structures. Then there is an event model $\mathcal{E}$ such that $\mathcal{M}_1 \oplus \mathcal{E} \leftrightarrows \mathcal{M}_2$ if and only if there is a total simulation from $\mathcal{M}_2$ to $\mathcal{M}_1$.*

Moreover, we define an efficient updating algorithm on knowledge structures that corresponds to the localized $\oplus$-update on epistemic models. As above, let $S = \wp(\mathsf{At})$: For every natural n, let S_n denote the set of worlds of the epistemic model $\langle \mathcal{F}_n(S), \{\sim_i\}_{i \in N}, V \rangle$. We define products $\times_n : \mathcal{F}_\alpha(S) \times \mathcal{F}_\alpha(S_n) \to \mathcal{F}_\alpha(S)$. These products correspond to the various possible updates with event models.

In order to state which models can be obtained by updates we need the following definition:

For a type $f \in \mathcal{F}_\alpha(S)$ we say that a type g is **admissible** for f iff

- $f_0 = g_0$
- for all agents i: $g_1(i) \subseteq f_1(i)$
- for $\alpha > 1$: If $h \in g_\alpha(i)$ then there is some $h' \in f_\alpha(i)$ such that h is admissible for h'

Our characterization theorem is similar to Theorem 1:

Theorem 2 *Let $f, g \in \mathcal{F}_\alpha(S)$ be types. Then g is obtainable by an update from f, i.e. there is some n and some $h \in \mathcal{F}_\alpha(S_n)$ such that $f \times_n h = g$ if and only if g is admissible for f.*

Further research is aimed at classifying what happens if we allow only updating types from a certain subclass of $\mathcal{F}_\alpha(S_n)$, for example those belonging to finite epistemic submodels $\langle \mathcal{F}_\alpha(S_n), \{\sim_i\}_{i \in N}, V \rangle$.

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[8]The idea to apply product update between Kripke structures has been studied by van Eijck et al. (2010). Our approach differs in a technical, but crucial, way (see the full paper for details). In particular, this means that our operation is not associative. Note that this should not be surprising given that we see a similar phenomena in the belief merging literature (cf. Maynard-Reid II and Shoham, 1998, Section 5.1).

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