Logics of Rational Agency

Lecture 1

Eric Pacuit

Tilburg Institute for Logic and Philosophy of Science
Tilburg Univeristy
ai.stanford.edu/~epacuit

June 21, 2010

Part 1: Introduction, Motivation and Background

Part 2: Ingredients of a Logical Analysis of Rational Agency

- Logics of Informational Attitudes and Informative Actions
- Logics of Motivational Attitudes (Preferences)
- Time, Action and Agency

Part 3: Merging Logics of Rational Agency

- Two Models of Information Dynamics
- "Epistemizing" Logics of Action and Ability: *knowing how to achieve* φ vs. *knowing that you can achieve* φ
- Entangling Knowledge/Beliefs and Preferences
- Preferences Change, Plans Change
- Logics of Rational Agency in Action: Logic and Games

Part 1: Introduction, Motivation and Background

Part 2: Ingredients of a Logical Analysis of Rational Agency

- Logics of Informational Attitudes and Informative Actions
- Logics of Motivational Attitudes (Preferences)
- Time, Action and Agency

Part 3: Merging Logics of Rational Agency

- Two Models of Information Dynamics
- "Epistemizing" Logics of Action and Ability: *knowing how to achieve* φ vs. *knowing that you can achieve* φ
- Entangling Knowledge/Beliefs and Preferences
- Preferences Change, Plans Change
- Logics of Rational Agency in Action: Logic and Games

Part 1: Introduction, Motivation and Background

Part 2: Ingredients of a Logical Analysis of Rational Agency

- Logics of Informational Attitudes and Informative Actions
- Logics of Motivational Attitudes (Preferences)
- Time, Action and Agency

Part 3: Merging Logics of Rational Agency

- Two Models of Information Dynamics
- "Epistemizing" Logics of Action and Ability: *knowing how to achieve* φ vs. *knowing that you can achieve* φ
- Entangling Knowledge/Beliefs and Preferences
- Preferences Change, Plans Change
- Logics of Rational Agency in Action: Logic and Games

Part 1: Introduction, Motivation and Background

Part 2: Ingredients of a Logical Analysis of Rational Agency

- Logics of Informational Attitudes and Informative Actions
- Logics of Motivational Attitudes (Preferences)
- Time, Action and Agency

Part 3: Merging Logics of Rational Agency

- Two Models of Information Dynamics
- "Epistemizing" Logics of Action and Ability: *knowing how to achieve* φ vs. *knowing that you can achieve* φ
- Entangling Knowledge/Beliefs and Preferences
- Preferences Change, Plans Change
- Logics of Rational Agency in Action: Logic and Games

Part 1: Introduction, Motivation and Background

Part 2: Ingredients of a Logical Analysis of Rational Agency

- Logics of Informational Attitudes and Informative Actions
- Logics of Motivational Attitudes (Preferences)
- Time, Action and Agency

Part 3: Merging Logics of Rational Agency

- Two Models of Information Dynamics
- "Epistemizing" Logics of Action and Ability: *knowing how to achieve* φ vs. *knowing that you can achieve* φ
- Entangling Knowledge/Beliefs and Preferences
- Preferences Change, Plans Change
- Logics of Rational Agency in Action: Logic and Games

Part 1: Introduction, Motivation and Background

Part 2: Ingredients of a Logical Analysis of Rational Agency

- Logics of Informational Attitudes and Informative Actions
- Logics of Motivational Attitudes (Preferences)
- Time, Action and Agency

Part 3: Merging Logics of Rational Agency

- Two Models of Information Dynamics
- "Epistemizing" Logics of Action and Ability: *knowing how to achieve* φ vs. *knowing that you can achieve* φ
- Entangling Knowledge/Beliefs and Preferences
- Preferences Change, Plans Change
- Logics of Rational Agency in Action: Logic and Games

Course Website

ai.stanford.edu/~epacuit/lograt/nasslli2010.html

Reading Material

✓ Pointers to literature on the website

Concerning Modal Logic

✓ Course by Blackburn and Areces

✓ *Modal Logic for Open Minds* by Johan van Benthem (CSLI, 2010)

Part 1: Introduction, Motivation and Background

We are interested in reasoning about rational (and not-so rational) agents engaged in some form of social interaction.

We are interested in reasoning about rational (and not-so rational) agents engaged in some form of social interaction.

- Philosophy (social epistemology, philosophy of action)
- Game Theory
- Social Choice Theory
- AI (multiagent systems)

We are interested in reasoning about rational (and not-so rational) agents engaged in some form of social interaction.

What is a "rational agent"? What *are we modeling?*

We are interested in reasoning about rational (and not-so rational) agents engaged in some form of social interaction.

What is a "rational agent"? What *are we modeling?*

- has consistent preferences (complete, transitive)
- (acts as if she) maximizes expected utility
- reacts to observations
- revises beliefs when learning a *surprising* piece of information
- understands higher-order information
- plans for the future
- asks questions
- ????

We are interested in reasoning about rational (and not-so rational) agents engaged in some form of social interaction.

- playing a (card) game
- having a conversation
- executing a *social procedure*
- ....

Goal: incorporate/extend existing game-theoretic/social choice analyses

We are interested in reasoning about rational (and not-so rational) agents engaged in some form of social interaction.

There is a jungle of logical frameworks!

- logics of informational attitudes (knowledge, beliefs, certainty)
- logics of action & agency
- temporal logics/dynamic logics
- logics of motivational attitudes (preferences, intentions)
- deontic logics

(*Not to mention various game-theoretic/social choice models and logical languages for reasoning about them*)

We are interested in reasoning about rational (and not-so rational) agents engaged in some form of social interaction.

- How can we compare different logical frameworks addressing similar aspects of rational agency and social interaction?
- How should we combine logical systems which address different aspects of social interaction towards the goal of a comprehensive (formal) theory of rational agency?
- How does a logical analysis contribute to the broader discussion of rational agency and social interaction within philosophy and the social sciences?

Game Theory

Game Theory

"We wish to find the mathematically complete principles which define 'rational behavior' for the participants." (pg. 31)

J. von Neumann and O. Morgenstern. *Theory of Games and Economic Behavior*. Princeton University Press, 1944.

Game Theory

"We wish to find the mathematically complete principles which define 'rational behavior' for the participants." (pg. 31)

J. von Neumann and O. Morgenstern. *Theory of Games and Economic Behavior*. Princeton University Press, 1944.

"Game theory is a bag of analytical tools designed to help us understand the phenomena that we observe when decision-makers interact." (pg. 1)

M. Osborne and A. Rubinstein. *Introduction to Game Theory*. MIT Press, 2004.

Game Situations

Bob

		L	R
Ann	U	1,2	0,0
	D	0,0	2,1

1. a group of *self-interested* agents (players) involved in some interdependent decision problem, and
2. the players *recognize that they are engaged in a game situation*

Game Situations

Bob

		L	R
Ann	U	1,2	0,0
	D	0,0	2,1

What should Ann (Bob) *do*?

Game Situations

Ann \ Bob	L	R
U	1,2	0,0
D	0,0	2,1

What does it mean for Ann to be *perfectly rational*?

Game Situations

Bob

Ann		L	R
	U	1,2	0,0
	D	0,0	2,1

Ann's best choice depends on what she *expects* Bob to do, and this depends on what she *thinks* Bob expects her to do, and so on...

Who is game theory about?

Who is game theory about?

1. **Classical view**: idealized world with *perfectly rational agents*

2. **Humanistic view**: real people in interactive situations

L. Samuelson. *Comments on Game Theory*. Game Theory: 5 Questions, Automatic Press, 2007.

Who is game theory about?

1. **Classical view**: idealized world with *perfectly rational agents*
    - The game itself it taken to be a literal description of the strategic interaction

"We adhere to the classical point of view that the game under consideration fully describes the real situation — that any (pre) commitment possibilities, any repetitive aspect, any probabilities of error, or any possibility of jointly observing some random event, have already been modeled in the game tree." (pg. 1005)

E. Kohlberg and J.-F. Mertens. *On the strategic stability of equilibria*. Econometrica, 54, pgs. 1003 - 1038, 1986.

L. Samuelson. *Comments on Game Theory*. Game Theory: 5 Questions, Automatic Press, 2007.

Who is game theory about?

1. **Classical view**: idealized world with *perfectly rational agents*
   - The game itself it taken to be a literal description of the strategic interaction
   - Any appropriate concept of equilibrium should be an *implication* of the information provided in the modeled interpreted through an assumption of perfect rationality.
2. **Humanistic view**: real people in interactive situations

L. Samuelson. *Comments on Game Theory*. Game Theory: 5 Questions, Automatic Press, 2007.

Who is game theory about?

1. **Classical view**: idealized world with *perfectly rational agents*
   - The game itself it taken to be a literal description of the strategic interaction
   - Any appropriate concept of equilibrium should be an *implication* of the information provided in the modeled interpreted through an assumption of perfect rationality.
2. **Humanistic view**: real people in interactive situations
   - the mathematical structures are *models* of interactive situations
   - the appropriate notion of equilibrium is part of the specification of the model

L. Samuelson. *Comments on Game Theory*. Game Theory: 5 Questions, Automatic Press, 2007.

But, the game models are missing something...

Formally, a game is described by its strategy sets and payoff functions.

But, the game models are missing something...

Formally, a game is described by its strategy sets and payoff functions. But in real life, may other parameters are relevant; there is a lot more going on. Situations that substantively are vastly different may nevertheless correspond to precisely the same strategic game.

But, the game models are missing something...

Formally, a game is described by its strategy sets and payoff functions. But in real life, may other parameters are relevant; there is a lot more going on. Situations that substantively are vastly different may nevertheless correspond to precisely the same strategic game. For example, in a parliamentary democracy with three parties, the winning coalitions are the same whether the parties hold a third of the seats, or, say, 49%, 39%, and 12 % respectively.

But, the game models are missing something...

Formally, a game is described by its strategy sets and payoff functions. But in real life, may other parameters are relevant; there is a lot more going on. Situations that substantively are vastly different may nevertheless correspond to precisely the same strategic game. For example, in a parliamentary democracy with three parties, the winning coalitions are the same whether the parties hold a third of the seats, or, say, 49%, 39%, and 12 % respectively. But the political situations are quite different.

But, the game models are missing something...

Formally, a game is described by its strategy sets and payoff functions. But in real life, may other parameters are relevant; there is a lot more going on. Situations that substantively are vastly different may nevertheless correspond to precisely the same strategic game. For example, in a parliamentary democracy with three parties, the winning coalitions are the same whether the parties hold a third of the seats, or, say, 49%, 39%, and 12 % respectively. But the political situations are quite different. The difference lies in the attitudes of the players, in their expectations about each other, in custom, and in history, though the rules of the game do not distinguish between the two situations. (pg. 72)

R. Aumann and J. H. Dreze. *Rational Expectation in Games*. American Economic Review, 98, pgs. 72-86, 2008.

What about a logical analysis?

What about a logical analysis?

- Which aspects of social situations should we focus on? Knowledge, Beliefs, Group Knowledge, Preferences, Desires, Ability, Actions, Intentions, Goals, Obligations, etc.

What about a logical analysis?

- Which aspects of social situations should we focus on? Knowledge, Beliefs, Group Knowledge, Preferences, Desires, Ability, Actions, Intentions, Goals, Obligations, etc.
- One grand system, or many smaller systems that loosely "fit" together?

What about a logical analysis?

- Which aspects of social situations should we focus on? Knowledge, Beliefs, Group Knowledge, Preferences, Desires, Ability, Actions, Intentions, Goals, Obligations, etc.
- One grand system, or many smaller systems that loosely "fit" together?
- *Combining* systems is hard! (conceptually and technically)

What about a logical analysis?

- Which aspects of social situations should we focus on? Knowledge, Beliefs, Group Knowledge, Preferences, Desires, Ability, Actions, Intentions, Goals, Obligations, etc.
- One grand system, or many smaller systems that loosely "fit" together?
- *Combining* systems is hard! (conceptually and technically)
- Logics *of* rational agents in social situations.

  vs.

  Logics *about* rational agents in social situations.

What about a logical analysis?

- Which aspects of social situations should we focus on? Knowledge, Beliefs, Group Knowledge, Preferences, Desires, Ability, Actions, Intentions, Goals, Obligations, etc.
- One grand system, or many smaller systems that loosely "fit" together?
- *Combining* systems is hard! (conceptually and technically)
- Logics *of* rational agents in social situations.

  vs.

  Logics *about* rational agents in social situations.
- Normative vs. Descriptive

The point of view of this model is not normative; it is not meant to advise the players what to do. The players do whatever they do; their strategies are taken as given.

The point of view of this model is not normative; it is not meant to advise the players what to do. The players do whatever they do; their strategies are taken as given. Neither is it meant as a description of what human beings actually do in interactive situations.

The point of view of this model is not normative; it is not meant to advise the players what to do. The players do whatever they do; their strategies are taken as given. Neither is it meant as a description of what human beings actually do in interactive situations. The most appropriate term is perhaps "analytic"; it asks, what are the implications *of rationality in interactive situations? Where does it lead?*

The point of view of this model is not normative; it is not meant to advise the players what to do. The players do whatever they do; their strategies are taken as given. Neither is it meant as a description of what human beings actually do in interactive situations. The most appropriate term is perhaps "analytic"; it asks, what are the implications *of rationality in interactive situations? Where does it lead? This question may be as important as, or even more important than, more direct "tests" of the relevance of the rationality hypothesis.* (pg. 622)

R. Aumann. *Irrationality in Game Theory*. in: *Aumann's Collected Papers, Volume 1*, Chapter 35, 1992.

Background

- Basic Modal Logic
- Weak Systems of Modal Logic
- Combining Logics
- Comparing Logics

Background

✓ Basic Modal Logic

⇒ Weak Systems of Modal Logic

⇒ Combining Logics

⇒ Comparing Logics

Weak Systems of Modal Logic

The Basic Modal Language: $\mathcal{L}$

$$p \mid \neg\varphi \mid \varphi \wedge \psi \mid \Box\varphi \mid \Diamond\varphi$$

where p is an atomic proposition (At)

Kripke (Relational) Models

$$\mathbb{M} = \langle W, R, V \rangle$$

Kripke (Relational) Models

$$\mathbb{M} = \langle W, R, V \rangle$$

- $W \neq \emptyset$
- $R \subseteq W \times W$
- $V : \mathsf{At} \rightarrow \wp(W)$

Truth in a Kripke Model

1. $\mathbb{M}, w \models p$ iff $w \in V(p)$
2. $\mathbb{M}, w \models \neg\varphi$ iff $\mathbb{M}, w \not\models \varphi$
3. $\mathbb{M}, w \models \varphi \wedge \psi$ iff $\mathbb{M}, w \models \varphi$ and $\mathbb{M}, w \models \psi$
4. $\mathbb{M}, w \models \Box\varphi$ iff for each $v \in W$, if wRv then $\mathbb{M}, v \models \varphi$
5. $\mathbb{M}, w \models \Diamond\varphi$ iff there is a $v \in W$ such that wRv and $\mathbb{M}, v \models \varphi$

Some Validities

(M) $\Box(\varphi \wedge \psi) \rightarrow \Box\varphi \wedge \Box\psi$

(C) $\Box\varphi \wedge \Box\psi \rightarrow \Box(\varphi \wedge \psi)$

(N) $\Box\top$

(K) $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$

(Dual) $\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$

(Nec) from $\vdash \varphi$ infer $\vdash \Box\varphi$

(Re) from $\vdash \varphi \leftrightarrow \psi$ infer $\vdash \Box\varphi \leftrightarrow \Box\psi$

Some Validities

(M) $\Box(\varphi \wedge \psi) \rightarrow \Box\varphi \wedge \Box\psi$

(Mon) $\dfrac{\vdash \varphi \rightarrow \psi}{\vdash \Box\varphi \rightarrow \Box\psi}$

(C) $\Box\varphi \wedge \Box\psi \rightarrow \Box(\varphi \wedge \psi)$

(N) $\Box\top$

(K) $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$

(Dual) $\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$

(Nec) from $\vdash \varphi$ infer $\vdash \Box\varphi$

(Re) from $\vdash \varphi \leftrightarrow \psi$ infer $\vdash \Box\varphi \leftrightarrow \Box\psi$

Neighborhoods

In a topology, a *neighborhood* of a point x is any set A containing x such that you can "wiggle" x without leaving A.

A *neighborhood system* of a point x is the collection of neighborhoods of x.

Neighborhoods in Modal Logic

Neighborhood Structure: $\langle W, N, V \rangle$

- $W \neq \emptyset$
- $N : W \rightarrow \wp(\wp(W))$
- $V : \mathsf{At} \rightarrow \wp(W)$

Some Notation

Given $\varphi \in \mathcal{L}$ and a model $\mathbb{M}$, the

- *proposition* expressed by φ
- *extension* of φ
- *truth set* of φ

is

Some Notation

Given $\varphi \in \mathcal{L}$ and a model $\mathbb{M}$, the

- *proposition* expressed by φ
- *extension* of φ
- *truth set* of φ

is

$$(\varphi)^{\mathbb{M}} = \{w \in W \mid \mathbb{M}, w \models \varphi\}$$

$w \models \Box\varphi$ if the truth set of φ is a neighborhood of w

$w \models \Box\varphi$ if the truth set of φ is a neighborhood of w

What does it mean to be a neighborhood?

$w \models \Box\varphi$ if the truth set of φ is a neighborhood of w

neighborhood in some topology.

J. McKinsey and A. Tarski. *The Algebra of Topology*. 1944.

$w \models \Box\varphi$ if the truth set of φ is a neighborhood of w

neighborhood in some topology.

J. McKinsey and A. Tarski. *The Algebra of Topology*. 1944.

contains all the immediate neighbors in some graph

S. Kripke. *A Semantic Analysis of Modal Logic*. 1963.

$w \models \Box\varphi$ if the truth set of φ is a neighborhood of w

neighborhood in some topology.

J. McKinsey and A. Tarski. *The Algebra of Topology*. 1944.

contains all the immediate neighbors in some graph

S. Kripke. *A Semantic Analysis of Modal Logic*. 1963.

an element of some distinguished collection of sets

D. Scott. *Advice on Modal Logic*. 1970.

R. Montague. *Pragmatics*. 1968.

To see the necessity of the more general approach, we could consider probability operators, conditional necessity, or, to invoke an especially perspicuous example of Dana Scott, the present progressive tense....Thus N might receive the awkward reading 'it is being the case that', in the sense in which 'it is being the case that Jones leaves' is synonymous with 'Jones is leaving'.

(Montague, pg. 73)

R. Montague. *Pragmatics and Intentional Logic*. 1970.

Segerberg's Essay

K. Segerberg. *An Essay on Classical Modal Logic*. Uppsula Technical Report, 1970.

Segerberg's Essay

K. Segerberg. *An Essay on Classical Modal Logic*. Uppsula Technical Report, 1970.

This essay purports to deal with classical modal logic. The qualification "classical" has not yet been given an established meaning in connection with modal logic....Clearly one would like to reserve the label "classical" for a category of modal logics which—if possible—is large enough to contain all or most of the systems which for historical or theoretical reasons have come to be regarded as important, and which also posses a high degree of naturalness and homogeneity.

(pg. 1)

Example: Logics of High Probability

$\Box\varphi$ means "φ *is assigned 'high' probability*", where *high* means above some threshold $r \in [0, 1]$.

Example: Logics of High Probability

$\Box\varphi$ means "φ *is assigned 'high' probability*", where *high* means above some threshold $r \in [0, 1]$.

Claim: Mon is a valid rule of inference.

Example: Logics of High Probability

$\Box\varphi$ means "*φ is assigned 'high' probability*", where *high* means above some threshold $r \in [0, 1]$.

Claim: Mon is a valid rule of inference.

Claim: C is not valid.

Example: Logics of High Probability

$\Box\varphi$ means "φ *is assigned 'high' probability*", where *high* means above some threshold $r \in [0, 1]$.

Claim: Mon is a valid rule of inference.

Claim: C is not valid.

H. Kyburg and C.M. Teng. *The Logic of Risky Knowledge*. Proceedings of WoLLIC (2002).

A. Herzig. *Modal Probability, Belief, and Actions*. Fundamenta Informaticae (2003).

Example: Social Choice Theory

$\Box\alpha$ mean "*the group accepts* α."

Example: Social Choice Theory

$\Box\alpha$ mean "*the group accepts* α."

Note: the language is restricted so that $\Box\Box\alpha$ *is not a wff.*

Example: Social Choice Theory

$\Box\alpha$ mean "*the group accepts* α."

Consensus: α is accepted provided *everyone* accepts α.

(E) $\Box\alpha \leftrightarrow \Box\beta$ provided $\alpha \leftrightarrow \beta$ is a tautology

(M) $\Box(\alpha \wedge \beta) \rightarrow (\Box\alpha \wedge \Box\beta)$

(C) $(\Box\alpha \wedge \Box\beta) \rightarrow (\Box\alpha \wedge \Box\beta)$

(N) $\Box\top$

(D) $\neg\Box\bot$

Example: Social Choice Theory

$\Box\alpha$ mean "*the group accepts* α."

Consensus: α is accepted provided *everyone* accepts α.

(E) $\Box\alpha \leftrightarrow \Box\beta$ provided $\alpha \leftrightarrow \beta$ is a tautology

(M) $\Box(\alpha \wedge \beta) \rightarrow (\Box\alpha \wedge \Box\beta)$

(C) $(\Box\alpha \wedge \Box\beta) \rightarrow (\Box\alpha \wedge \Box\beta)$

(N) $\Box\top$

(D) $\neg\Box\bot$

Theorem The above axioms axiomatize consensus (provided $n \geq 2^{|At|}$)).

Example: Social Choice Theory

$\Box\alpha$ mean "*the group accepts* α."

Majority: α is accepted if a *majority* of the agents accept α.

Example: Social Choice Theory

$\Box\alpha$ mean "*the group accepts* α."

Majority: α is accepted if a *majority* of the agents accept α.

(E) $\Box\alpha \leftrightarrow \Box\beta$ provided $\alpha \leftrightarrow \beta$ is a tautology

(M) $\Box(\alpha \wedge \beta) \rightarrow (\Box\alpha \wedge \Box\beta)$

(S) $\Box\alpha \rightarrow \neg\Box\neg\alpha$

(T) $([\geq]\varphi_1 \wedge \cdots \wedge [\geq]\varphi_k \wedge [\leq]\psi_1 \wedge \cdots \wedge [\leq]\psi_k) \rightarrow \bigwedge_{1\leq i\leq k}([=]\varphi_i \wedge [=]\psi_i)$ where $\forall v \in V_I$: $|\{i \mid v_(\varphi_i) = 1\}| = |\{i \mid v(\psi_i) = 1\}|$

Theorem The above axioms axiomatize majority.

Example: Social Choice Theory

$\Box\alpha$ mean "*the group accepts* α."

M. Pauly. *Axiomatizing Collective Judgement Sets in a Minimal Logical Language*. 2006.

T. Daniëls. *Social Choice and Logic via Simple Games*. ILLC, Masters Thesis, 2007.

Many Other Examples

- Epistemic Logic: the logical omniscience problem.

M. Vardi. *On Epistemic Logic and Logical Omniscience*. TARK (1986).

- Reasoning about coalitions

M. Pauly. *Logic for Social Software*. Ph.D. Thesis, ILLC (2001).

- Knowledge Representation

V. Padmanabhan, G. Governatori, K. Su . *Knowledge Assesment: A Modal Logic Approach*. KRAQ (2007).

- Program logics: modeling concurrent programs

D. Peleg. *Concurrent Dynamic Logic*. J. ACM (1987).

- More during this course....

Neighborhood Frames

Let W be a non-empty set of states.

Any map $N : W \to \wp(\wp(W))$ is called a neighborhood function

Definition

A pair $\langle W, N \rangle$ is a called a neighborhood frame if W a non-empty set and N is a neighborhood function.

Some Terminology

Let $\mathcal{F} = \langle W, N \rangle$ be a neighborhood frame.

- $\mathcal{F}$ is closed under intersections if for any collections of sets $\{X_i\}_{i \in I}$ such that for each $i \in I$, $X_i \in N(w)$, then $\cap_{i \in I} X_i \in N(w)$.
- $\mathcal{F}$ is supplemented, or closed under supersets or monotonic provided for each $X \subseteq W$, if $X \in N(w)$ and $X \subseteq Y \subseteq W$, then $Y \in N(w)$.
- $\mathcal{F}$ contains the unit provided $W \in N(w)$
- the set $\cap_{X \in \mathcal{F}} X$ the core of $\mathcal{F}$. $\mathcal{F}$ contains its core provided $\cap_{X \in \mathcal{F}} X \in \mathcal{F}$.
- $\mathcal{F}$ is augmented if $\mathcal{F}$ contains it core and is supplemented.

Some Terminology

Let $\mathcal{F} = \langle W, N \rangle$ be a neighborhood frame.

- $\mathcal{F}$ is closed under intersections if for any collections of sets $\{X_i\}_{i \in I}$ such that for each $i \in I$, $X_i \in N(w)$, then $\cap_{i \in I} X_i \in N(w)$.
- $\mathcal{F}$ is supplemented, or closed under supersets or monotonic provided for each $X \subseteq W$, if $X \in N(w)$ and $X \subseteq Y \subseteq W$, then $Y \in N(w)$.
- $\mathcal{F}$ contains the unit provided $W \in N(w)$
- the set $\cap_{X \in \mathcal{F}} X$ the core of $\mathcal{F}$. $\mathcal{F}$ contains its core provided $\cap_{X \in \mathcal{F}} X \in \mathcal{F}$.
- $\mathcal{F}$ is augmented if $\mathcal{F}$ contains it core and is supplemented.

Some Terminology

Let $\mathcal{F} = \langle W, N \rangle$ be a neighborhood frame.

- $\mathcal{F}$ is closed under intersections if for any collections of sets $\{X_i\}_{i \in I}$ such that for each $i \in I$, $X_i \in N(w)$, then $\cap_{i \in I} X_i \in N(w)$.
- $\mathcal{F}$ is supplemented, or closed under supersets or monotonic provided for each $X \subseteq W$, if $X \in N(w)$ and $X \subseteq Y \subseteq W$, then $Y \in N(w)$.
- $\mathcal{F}$ contains the unit provided $W \in N(w)$
- the set $\cap_{X \in \mathcal{F}} X$ the core of $\mathcal{F}$. $\mathcal{F}$ contains its core provided $\cap_{X \in \mathcal{F}} X \in \mathcal{F}$.
- $\mathcal{F}$ is augmented if $\mathcal{F}$ contains it core and is supplemented.

Some Terminology

Let $\mathcal{F} = \langle W, N \rangle$ be a neighborhood frame.

- $\mathcal{F}$ is closed under intersections if for any collections of sets $\{X_i\}_{i \in I}$ such that for each $i \in I$, $X_i \in N(w)$, then $\cap_{i \in I} X_i \in N(w)$.
- $\mathcal{F}$ is supplemented, or closed under supersets or monotonic provided for each $X \subseteq W$, if $X \in N(w)$ and $X \subseteq Y \subseteq W$, then $Y \in N(w)$.
- $\mathcal{F}$ contains the unit provided $W \in N(w)$
- the set $\cap_{X \in \mathcal{F}} X$ the core of $\mathcal{F}$. $\mathcal{F}$ contains its core provided $\cap_{X \in \mathcal{F}} X \in \mathcal{F}$.
- $\mathcal{F}$ is augmented if $\mathcal{F}$ contains it core and is supplemented.

Some Terminology

Let $\mathcal{F} = \langle W, N \rangle$ be a neighborhood frame.

- $\mathcal{F}$ is closed under intersections if for any collections of sets $\{X_i\}_{i \in I}$ such that for each $i \in I$, $X_i \in N(w)$, then $\cap_{i \in I} X_i \in N(w)$.
- $\mathcal{F}$ is supplemented, or closed under supersets or monotonic provided for each $X \subseteq W$, if $X \in N(w)$ and $X \subseteq Y \subseteq W$, then $Y \in N(w)$.
- $\mathcal{F}$ contains the unit provided $W \in N(w)$
- the set $\cap_{X \in \mathcal{F}} X$ the core of $\mathcal{F}$. $\mathcal{F}$ contains its core provided $\cap_{X \in \mathcal{F}} X \in \mathcal{F}$.
- $\mathcal{F}$ is augmented if $\mathcal{F}$ contains it core and is supplemented.

From Kripke Frames to Neighborhood Frames

Let $R \subseteq W \times W$, define a map $R^{\rightarrow} : W \rightarrow \wp W$:

$$\text{for each } w \in W, \text{ let } R^{\rightarrow}(w) = \{v \mid wRv\}$$

From Kripke Frames to Neighborhood Frames

Let $R \subseteq W \times W$, define a map $R^{\rightarrow} : W \rightarrow \wp W$:

$$\text{for each } w \in W, \text{ let } R^{\rightarrow}(w) = \{v \mid wRv\}$$

Definition

Given a relation R on a set W and a state $w \in W$. A set $X \subseteq W$ is *R-necessary at w* if $R^{\rightarrow}(w) \subseteq X$.

From Kripke Frames to Neighborhood Frames

Let $R \subseteq W \times W$, define a map $R^{\rightarrow} : W \rightarrow \wp W$:

$$\text{for each } w \in W, \text{ let } R^{\rightarrow}(w) = \{v \mid wRv\}$$

Let $\mathcal{N}_w^R$ be the set of sets that are R-necessary at w:

$$\mathcal{N}_w^R = \{X \mid R^{\rightarrow}(w) \subseteq X\}$$

From Kripke Frames to Neighborhood Frames

Let $R \subseteq W \times W$, define a map $R^{\rightarrow} : W \rightarrow \wp W$:

$$\text{for each } w \in W, \text{ let } R^{\rightarrow}(w) = \{v \mid wRv\}$$

Let $\mathcal{N}_w^R$ be the set of sets that are R-necessary at w:

$$\mathcal{N}_w^R = \{X \mid R^{\rightarrow}(w) \subseteq X\}$$

Lemma

Let R be a relation on W. Then for each $w \in W$, $\mathcal{N}_w^R$ is augmented.

From Kripke Frames to Neighborhood Frames

Properties of R are reflected in $\mathcal{N}_w^R$:

- If R is reflexive, then for each $w \in W$, $w \in \cap\mathcal{N}_w^R$
- If R is transitive then for each $w \in W$, if $X \in \mathcal{N}_w^R$, then $\{v \mid X \in \mathcal{N}_v^R\} \in \mathcal{N}_w^R$.

From Neighborhood Frames to Kripke Frames

Theorem

- *Let $\langle W, R \rangle$ be a relational frame. Then there is an equivalent augmented neighborhood frame.*
- *Let $\langle W, N \rangle$ be an augmented neighborhood frame. Then there is an equivalent relational frame.*

From Neighborhood Frames to Kripke Frames

for all $X \subseteq W$, $X \in N(w)$ iff $X \in \mathcal{N}_w^R$.

Theorem

- *Let $\langle W, R \rangle$ be a relational frame. Then there is an equivalent augmented neighborhood frame.*
- *Let $\langle W, N \rangle$ be an augmented neighborhood frame. Then there is an equivalent relational frame.*

From Neighborhood Frames to Kripke Frames

Theorem

✓ *Let $\langle W, R\rangle$ be a relational frame. Then there is an equivalent augmented neighborhood frame.*

- *Let $\langle W, N\rangle$ be an augmented neighborhood frame. Then there is an equivalent relational frame.*

Proof.
For each $w \in W$, let $N(w) = \mathcal{N}_w^R$. □

From Neighborhood Frames to Kripke Frames

Theorem

- *Let $\langle W, R \rangle$ be a relational frame. Then there is an equivalent augmented neighborhood frame.*

✓ *Let $\langle W, N \rangle$ be an augmented neighborhood frame. Then there is an equivalent relational frame.*

Proof.

For each $w, v \in W$, $wR_N v$ iff $v \in \cap N(w)$. □

Neighborhood Model

Let $\mathfrak{F} = \langle W, N \rangle$ be a neighborhood frame. A neighborhood model based on $\mathfrak{F}$ is a tuple $\langle W, N, V \rangle$ where $V : \mathsf{At} \rightarrow 2^W$ is a valuation function.

Truth in a Model

- $\mathfrak{M}, w \models p$ iff $w \in V(p)$
- $\mathfrak{M}, w \models \neg\varphi$ iff $\mathfrak{M}, w \not\models \varphi$
- $\mathfrak{M}, w \models \varphi \wedge \psi$ iff $\mathfrak{M}, w \models \varphi$ and $\mathfrak{M}, w \models \psi$

Truth in a Model

- $\mathfrak{M}, w \models p$ iff $w \in V(p)$
- $\mathfrak{M}, w \models \neg\varphi$ iff $\mathfrak{M}, w \not\models \varphi$
- $\mathfrak{M}, w \models \varphi \wedge \psi$ iff $\mathfrak{M}, w \models \varphi$ and $\mathfrak{M}, w \models \psi$
- $\mathfrak{M}, w \models \Box\varphi$ iff $(\varphi)^{\mathfrak{M}} \in N(w)$
- $\mathfrak{M}, w \models \Diamond\varphi$ iff $W - (\varphi)^{\mathfrak{M}} \notin N(w)$

where $(\varphi)^{\mathfrak{M}} = \{w \mid \mathfrak{M}, w \models \varphi\}$.

Detailed Example

Suppose $W = \{w, s, v\}$ is the set of states and define a neighborhood model $\mathfrak{M} = \langle W, N, V\rangle$ as follows:

- $N(w) = \{\{s\}, \{v\}, \{w, v\}\}$
- $N(s) = \{\{w, v\}, \{w\}, \{w, s\}\}$
- $N(v) = \{\{s, v\}, \{w\}, \emptyset\}$

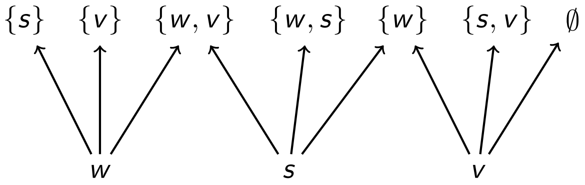
Further suppose that $V(p) = \{w, s\}$ and $V(q) = \{s, v\}$.

Detailed Example

Suppose $W = \{w, s, v\}$ is the set of states and define a neighborhood model $\mathfrak{M} = \langle W, N, V \rangle$ as follows:

- $N(w) = \{\{s\}, \{v\}, \{w, v\}\}$
- $N(s) = \{\{w, v\}, \{w\}, \{w, s\}\}$
- $N(v) = \{\{s, v\}, \{w\}, \emptyset\}$

Further suppose that $V(p) = \{w, s\}$ and $V(q) = \{s, v\}$.

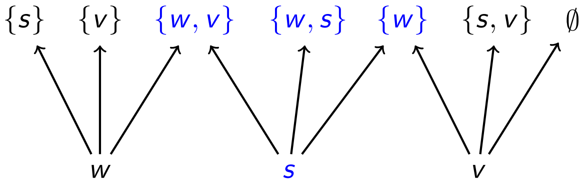


Detailed Example

Suppose $W = \{w, s, v\}$ is the set of states and define a neighborhood model $\mathfrak{M} = \langle W, N, V\rangle$ as follows:

- $N(w) = \{\{s\}, \{v\}, \{w, v\}\}$
- $N(s) = \{\{w, v\}, \{w\}, \{w, s\}\}$
- $N(v) = \{\{s, v\}, \{w\}, \emptyset\}$

Further suppose that $V(p) = \{w, s\}$ and $V(q) = \{s, v\}$.

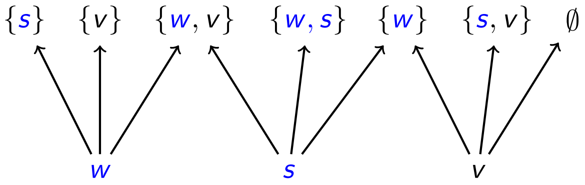


Detailed Example

Suppose $W = \{w, s, v\}$ is the set of states and define a neighborhood model $\mathfrak{M} = \langle W, N, V \rangle$ as follows:

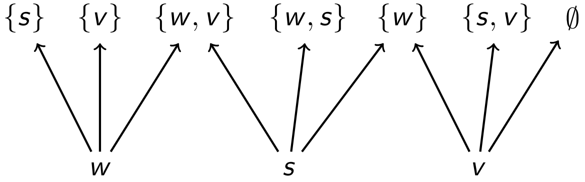
- $N(w) = \{\{s\}, \{v\}, \{w, v\}\}$
- $N(s) = \{\{w, v\}, \{w\}, \{w, s\}\}$
- $N(v) = \{\{s, v\}, \{w\}, \emptyset\}$

Further suppose that $V(p) = \{w, s\}$ and $V(q) = \{s, v\}$.



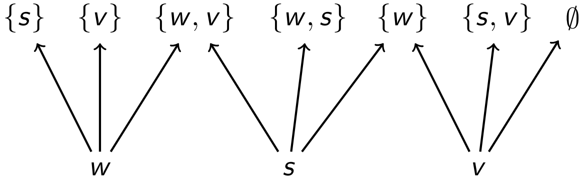
Detailed Example

$V(p) = \{w, s\}$ and $V(q) = \{s, v\}$



Detailed Example

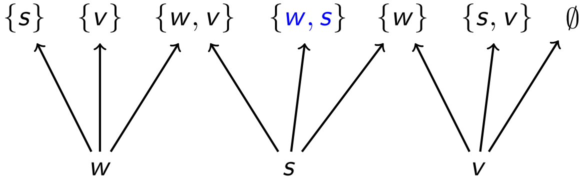
$V(p) = \{w, s\}$ and $V(q) = \{s, v\}$



$$\mathfrak{M}, s \models \Box p$$

Detailed Example

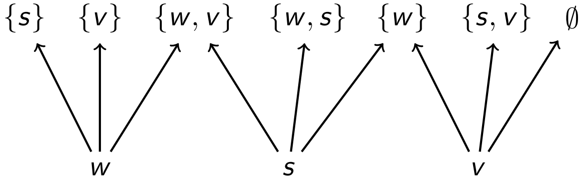
$V(p) = \{w, s\}$ and $V(q) = \{s, v\}$



$\mathfrak{M}, s \models \Box p$

Detailed Example

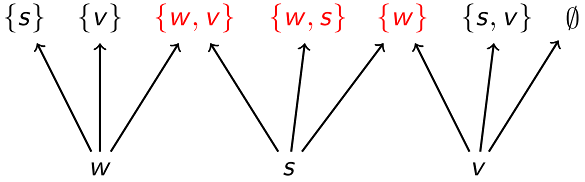
$V(p) = \{w, s\}$ and $V(q) = \{s, v\}$



$$\mathfrak{M}, s \models \Diamond p$$

Detailed Example

$V(p) = \{w, s\}$ and $V(q) = \{s, v\}$

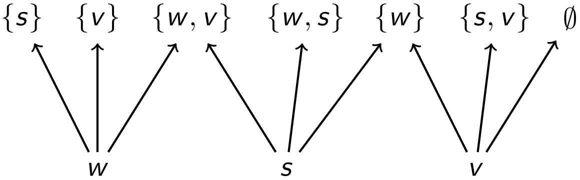


$$\mathfrak{M}, s \models \Diamond p$$

$$(\neg p)^{\mathfrak{M}} = \{v\}$$

Detailed Example

$V(p) = \{w, s\}$ and $V(q) = \{s, v\}$



$\mathfrak{M}, w \models \Diamond\Box p$?

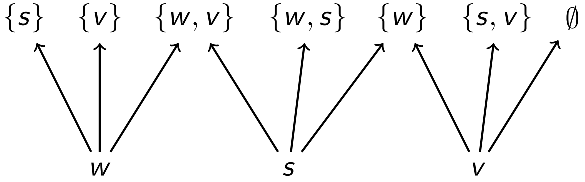
$\mathfrak{M}, w \models \Box\Box p$?

$\mathfrak{M}, v \models \Box\Diamond p$?

$\mathfrak{M}, v \models \Diamond\Box p$?

Detailed Example

$V(p) = \{w, s\}$ and $V(q) = \{s, v\}$



$\mathfrak{M}, w \models \Diamond\Box p$?

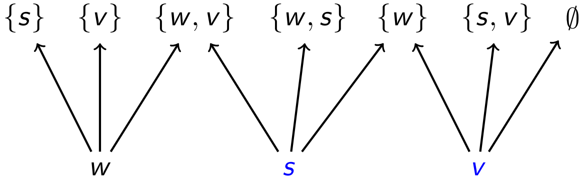
$\mathfrak{M}, w \models \Box\Box p$?

$\mathfrak{M}, v \models \Box\Diamond p$

$\mathfrak{M}, v \models \Diamond\Box p$?

Detailed Example

$V(p) = \{w, s\}$ and $V(q) = \{s, v\}$



$\mathfrak{M}, w \models \Diamond\Box p$?

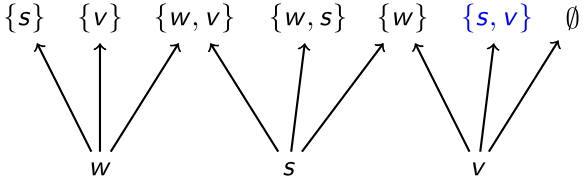
$\mathfrak{M}, w \models \Box\Box p$?

$\mathfrak{M}, v \models \Box\Diamond p$

$\mathfrak{M}, v \models \Diamond\Box p$?

Detailed Example

$V(p) = \{w, s\}$ and $V(q) = \{s, v\}$



$\mathfrak{M}, w \models \Diamond\Box p$?

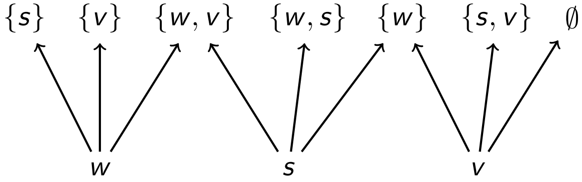
$\mathfrak{M}, w \models \Box\Box p$?

$\mathfrak{M}, v \models \Box\Diamond p$

$\mathfrak{M}, v \models \Diamond\Box p$?

Detailed Example

$V(p) = \{w, s\}$ and $V(q) = \{s, v\}$



$\mathfrak{M}, w \not\models \Diamond\Box p$

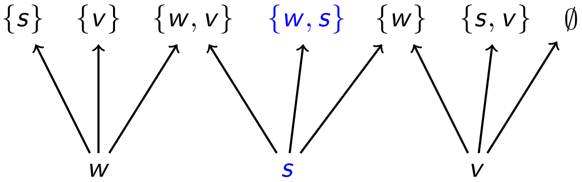
$\mathfrak{M}, w \models \Box\Box p$

$\mathfrak{M}, v \models \Box\Diamond p$

$\mathfrak{M}, v \models \Diamond\Box p$

Detailed Example

$V(p) = \{w, s\}$ and $V(q) = \{s, v\}$



$\mathfrak{M}, w \not\models \Diamond\Box p$

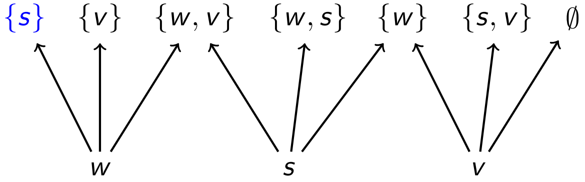
$\mathfrak{M}, w \models \Box\Box p$

$\mathfrak{M}, v \models \Box\Diamond p$

$\mathfrak{M}, v \models \Diamond\Box p$

Detailed Example

$V(p) = \{w, s\}$ and $V(q) = \{s, v\}$



$\mathfrak{M}, w \not\models \Diamond\Box p$

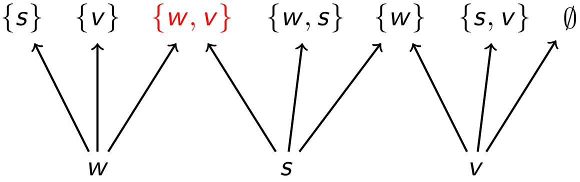
$\mathfrak{M}, w \models \Box\Box p$

$\mathfrak{M}, v \models \Box\Diamond p$

$\mathfrak{M}, v \models \Diamond\Box p$

Detailed Example

$V(p) = \{w, s\}$ and $V(q) = \{s, v\}$



$\mathfrak{M}, w \not\models \Diamond\Box p$

$\mathfrak{M}, w \models \Box\Box p$

$\mathfrak{M}, v \models \Box\Diamond p$

$\mathfrak{M}, v \models \Diamond\Box p$

What can we say?

Definition

A modal formula φ defines a property P of neighborhood functions if any neighborhood frame $\mathfrak{F}$ has property P iff $\mathfrak{F}$ validates φ.

What can we say?

Lemma

Let $\mathfrak{F} = \langle W, N \rangle$ be a neighborhood frame. Then $\mathfrak{F} \models \Box(\varphi \wedge \psi) \rightarrow \Box\varphi \wedge \Box\psi$ iff $\mathfrak{F}$ is closed under supersets.

What can we say?

Lemma

Let $\mathfrak{F} = \langle W, N \rangle$ *be a neighborhood frame. Then* $\mathfrak{F} \models \Box(\varphi \wedge \psi) \rightarrow \Box\varphi \wedge \Box\psi$ *iff* $\mathfrak{F}$ *is closed under supersets.*

Lemma

Let $\mathfrak{F} = \langle W, N \rangle$ *be a neighborhood frame. Then* $\mathfrak{F} \models \Box\varphi \wedge \Box\psi \rightarrow \Box(\varphi \wedge \psi)$ *iff* $\mathfrak{F}$ *is closed under finite intersections.*

What can we say?

Consider the formulas $\Diamond\top$ and $\Box\varphi \rightarrow \Diamond\varphi$.

What can we say?

Consider the formulas $\Diamond\top$ and $\Box\varphi \rightarrow \Diamond\varphi$.

On relational frames, these formulas both define the same property: seriality.

What can we say?

Consider the formulas $\Diamond\top$ and $\Box\varphi \rightarrow \Diamond\varphi$.

On relational frames, these formulas both define the same property: seriality.

On neighborhood frames:

- $\Diamond\top$ corresponds to the property $\emptyset \notin N(w)$

What can we say?

Consider the formulas $\Diamond\top$ and $\Box\varphi \to \Diamond\varphi$.

On relational frames, these formulas both define the same property: seriality.

On neighborhood frames:

- $\Diamond\top$ corresponds to the property $\emptyset \notin N(w)$
- $\Box\varphi \to \Diamond\varphi$ is valid on $\mathfrak{F}$ iff $\mathfrak{F}$ is proper.

What can we say?

Lemma

Let $\mathfrak{F} = \langle W, N \rangle$ *be a neighborhood frame such that for each* $w \in W$, $N(w) \neq \emptyset$.

1. $\mathfrak{F} \models \Box\varphi \rightarrow \varphi$ *iff for each* $w \in W$, $w \in \cap N(w)$
2. $\mathfrak{F} \models \Box\varphi \rightarrow \Box\Box\varphi$ *iff for each* $w \in W$, *if* $X \in N(w)$, *then* $\{v \mid X \in N(v)\} \in N(w)$

Find properties on frames that are defined by the following formulas:

1. $\Box\bot$
2. $\neg\Box\varphi \rightarrow \Box\neg\Box\varphi$
3. $\Diamond\varphi \rightarrow \Box\varphi$
4. $\Diamond\Box\varphi \rightarrow \Box\Diamond\varphi$
5. $\Box\Diamond\varphi \rightarrow \Diamond\Box\varphi$

Some Non-validities

1. $\Box(\varphi \wedge \psi) \rightarrow \Box\varphi \wedge \Box\psi$
2. $\Box\varphi \wedge \Box\psi \rightarrow \Box(\varphi \wedge \psi)$
3. $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$
4. $\Box\top$
5. $\Box\varphi \rightarrow \varphi$
6. $\Box\varphi \rightarrow \Box\Box\varphi$
7. Many more...

Validities

(Dual) $\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$ is valid in all neighborhood models.

(Re) If $\varphi \leftrightarrow \psi$ is valid then $\Box\varphi \leftrightarrow \Box\psi$ is valid.

Other modal operators

- $\mathfrak{M}, w \models \langle\ \rangle\varphi$ iff $\exists X \in N(w)$ such that $\exists v \in X$, $\mathfrak{M}, v \models \varphi$
- $\mathfrak{M}, w \models [\ ]\varphi$ iff $\forall X \in N(w)$ such that $\forall v \in X$, $\mathfrak{M}, v \models \varphi$

- $\mathfrak{M}, w \models \langle\ ]\varphi$ iff $\exists X \in N(w)$ such that $\forall v \in X$, $\mathfrak{M}, v \models \varphi$
- $\mathfrak{M}, w \models [\ \rangle\varphi$ iff $\forall X \in N(w)$ such that $\exists v \in X$, $\mathfrak{M}, v \models \varphi$

Other modal operators

- $\mathfrak{M}, w \models \langle\ \rangle\varphi$ iff $\exists X \in N(w)$ such that $\exists v \in X$, $\mathfrak{M}, v \models \varphi$
- $\mathfrak{M}, w \models [\ ]\varphi$ iff $\forall X \in N(w)$ such that $\forall v \in X$, $\mathfrak{M}, v \models \varphi$

- $\mathfrak{M}, w \models \langle\ ]\varphi$ iff $\exists X \in N(w)$ such that $\forall v \in X$, $\mathfrak{M}, v \models \varphi$
- $\mathfrak{M}, w \models [\ \rangle\varphi$ iff $\forall X \in N(w)$ such that $\exists v \in X$, $\mathfrak{M}, v \models \varphi$

Other modal operators

- $\mathfrak{M}, w \models \langle\ ]\varphi$ iff $\exists X \in N(w)$ such that $\forall v \in X$, $\mathfrak{M}, v \models \varphi$
- $\mathfrak{M}, w \models [\ \rangle\varphi$ iff $\forall X \in N(w)$ such that $\exists v \in X$, $\mathfrak{M}, v \models \varphi$

Other modal operators

- $\mathfrak{M}, w \models \langle\ ]\varphi$ iff $\exists X \in N(w)$ such that $\forall v \in X$, $\mathfrak{M}, v \models \varphi$
- $\mathfrak{M}, w \models [\ \rangle\varphi$ iff $\forall X \in N(w)$ such that $\exists v \in X$, $\mathfrak{M}, v \models \varphi$

Lemma

Let $\mathfrak{M} = \langle W, N, V\rangle$ be a neighborhood model. The for each $w \in W$,

1. *if $\mathfrak{M}, w \models \Box\varphi$ then $\mathfrak{M}, w \models \langle\ ]\varphi$*
2. *if $\mathfrak{M}, w \models [\ \rangle\varphi$ then $\mathfrak{M}, w \models \Diamond\varphi$*

However, the converses of the above statements are false.

Other modal operators

- $\mathfrak{M}, w \models \langle\ ]\varphi$ iff $\exists X \in N(w)$ such that $\forall v \in X$, $\mathfrak{M}, v \models \varphi$
- $\mathfrak{M}, w \models [\ \rangle\varphi$ iff $\forall X \in N(w)$ such that $\exists v \in X$, $\mathfrak{M}, v \models \varphi$

Lemma

1. *If* $\varphi \rightarrow \psi$ *is valid, then so is* $\langle\ ]\varphi \rightarrow \langle\ ]\psi$.
2. $\langle\ ](\varphi \wedge \psi) \rightarrow (\langle\ ]\varphi \wedge \langle\ ]\psi)$ *is valid*

Investigate analogous results for the other modal operators defined above.

Different Semantics

A multi-relational Kripke model is a triple $\mathbb{M} = \langle W, \mathcal{R}, V \rangle$ where $\mathcal{R} \subseteq \wp(W \times W)$.

Different Semantics

A multi-relational Kripke model is a triple $\mathbb{M} = \langle W, \mathcal{R}, V \rangle$ where $\mathcal{R} \subseteq \wp(W \times W)$.

$\mathbb{M}, w \models \Box\varphi$ iff $\exists R \in \mathcal{R}$ such that $\forall v \in W$, if wRv then $\mathbb{M}, v \models \varphi$.

Different Semantics

A multi-relational Kripke model is a triple $\mathbb{M} = \langle W, \mathcal{R}, V\rangle$ where $\mathcal{R} \subseteq \wp(W \times W)$.

$\mathbb{M}, w \models \Box\varphi$ iff $\exists R \in \mathcal{R}$ such that $\forall v \in W$, if wRv then $\mathbb{M}, v \models \varphi$.

Are multi-relational semantics *equivalent* to neighborhood semantics?

Different Semantics

A multi-relational Kripke model is a triple $\mathbb{M} = \langle W, \mathcal{R}, V \rangle$ where $\mathcal{R} \subseteq \wp(W \times W)$.

$\mathbb{M}, w \models \Box\varphi$ iff $\exists R \in \mathcal{R}$ such that $\forall v \in W$, if wRv then $\mathbb{M}, v \models \varphi$.

Are multi-relational semantics *equivalent* to neighborhood semantics? Almost

Different Semantics

A multi-relational Kripke model is a triple $\mathbb{M} = \langle W, \mathcal{R}, V\rangle$ where $\mathcal{R} \subseteq \wp(W \times W)$.

$\mathbb{M}, w \models \Box\varphi$ iff $\exists R \in \mathcal{R}$ such that $\forall v \in W$, if wRv then $\mathbb{M}, v \models \varphi$.

A world is called queer if nothing is necessary and everything is possible.

Different Semantics

A multi-relational Kripke model is a triple $\mathbb{M} = \langle W, \mathcal{R}, V \rangle$ where $\mathcal{R} \subseteq \wp(W \times W)$.

$\mathbb{M}, w \models \Box\varphi$ iff $\exists R \in \mathcal{R}$ such that $\forall v \in W$, if wRv then $\mathbb{M}, v \models \varphi$.

w is queer iff $N(w) = \emptyset$

Different Semantics

A multi-relational Kripke model is a triple $\mathbb{M} = \langle W, \mathcal{R}, V \rangle$ where $\mathcal{R} \subseteq \wp(W \times W)$.

$\mathbb{M}, w \models \Box\varphi$ iff $\exists R \in \mathcal{R}$ such that $\forall v \in W$, if wRv then $\mathbb{M}, v \models \varphi$.

w is queer iff $N(w) = \emptyset$

A multi-relational model with queer worlds is a quadruple $\mathbb{M} = \langle W, Q, \mathcal{R}, V \rangle$.

Different Semantics

A multi-relational Kripke model is a triple $\mathbb{M} = \langle W, \mathcal{R}, V\rangle$ where $\mathcal{R} \subseteq \wp(W \times W)$.

$\mathbb{M}, w \models \Box\varphi$ iff $\exists R \in \mathcal{R}$ such that $\forall v \in W$, if wRv then $\mathbb{M}, v \models \varphi$.

w is queer iff $N(w) = \emptyset$

A multi-relational model with queer worlds is a quadruple $\mathbb{M} = \langle W, Q, \mathcal{R}, V\rangle$.

$\mathbb{M}, w \models \Box\varphi$ iff $w \notin Q$ and $\exists R \in \mathcal{R}$ such that $\forall v \in W$, if wRv then $\mathbb{M}, v \models \varphi$.

Different Semantics

M. Fitting. *Proof Methods for Modal and Intuitionistic Logics*. 1983.

L. Goble. *Multiplex semantics for Deontic Logic*. Nordic Journal of Philosophical Logic (2000).

G. Governatori and A. Rotolo. *On the axiomatization of Elgesems logic of agency and ability*. JPL (2005).

PC Propositional Calculus

E $\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$

M $\Box(\varphi \wedge \psi) \rightarrow (\Box\varphi \wedge \Box\psi)$

C $(\Box\varphi \wedge \Box\psi) \rightarrow \Box(\varphi \wedge \psi)$

N $\Box\top$

K $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$

RE $\dfrac{\varphi \leftrightarrow \psi}{\Box\varphi \leftrightarrow \Box\psi}$

Nec $\dfrac{\varphi}{\Box\varphi}$

MP $\dfrac{\varphi \quad \varphi \rightarrow \psi}{\psi}$

PC Propositional Calculus

E $\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$

M $\Box(\varphi \wedge \psi) \rightarrow (\Box\varphi \wedge \Box\psi)$

C $(\Box\varphi \wedge \Box\psi) \rightarrow \Box(\varphi \wedge \psi)$

N $\Box\top$

K $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$

RE $\dfrac{\varphi \leftrightarrow \psi}{\Box\varphi \leftrightarrow \Box\psi}$

Nec $\dfrac{\varphi}{\Box\varphi}$

MP $\dfrac{\varphi \quad \varphi \rightarrow \psi}{\psi}$

A modal logic **L** is classical if it contains all instances of *E* and is closed under *RE*.

PC Propositional Calculus

E $\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$

M $\Box(\varphi \wedge \psi) \rightarrow (\Box\varphi \wedge \Box\psi)$

C $(\Box\varphi \wedge \Box\psi) \rightarrow \Box(\varphi \wedge \psi)$

N $\Box\top$

K $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$

RE $\dfrac{\varphi \leftrightarrow \psi}{\Box\varphi \leftrightarrow \Box\psi}$

Nec $\dfrac{\varphi}{\Box\varphi}$

MP $\dfrac{\varphi \quad \varphi \rightarrow \psi}{\psi}$

A modal logic **L** is classical if it contains all instances of *E* and is closed under *RE*.

E is the smallest classical modal logic.

PC Propositional Calculus

E $\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$

M $\Box(\varphi \wedge \psi) \rightarrow (\Box\varphi \wedge \Box\psi)$

C $(\Box\varphi \wedge \Box\psi) \rightarrow \Box(\varphi \wedge \psi)$

N $\Box\top$

K $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$

RE $\dfrac{\varphi \leftrightarrow \psi}{\Box\varphi \leftrightarrow \Box\psi}$

Nec $\dfrac{\varphi}{\Box\varphi}$

MP $\dfrac{\varphi \quad \varphi \rightarrow \psi}{\psi}$

E is the smallest classical modal logic.

In **E**, *M* is equivalent to

(*Mon*) $\dfrac{\varphi \rightarrow \psi}{\Box\varphi \rightarrow \Box\psi}$

PC Propositional Calculus

E $\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$

Mon $$\frac{\varphi \rightarrow \psi}{\Box\varphi \rightarrow \Box\psi}$$

C $(\Box\varphi \wedge \Box\psi) \rightarrow \Box(\varphi \wedge \psi)$

N $\Box\top$

K $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$

RE $$\frac{\varphi \leftrightarrow \psi}{\Box\varphi \leftrightarrow \Box\psi}$$

Nec $$\frac{\varphi}{\Box\varphi}$$

MP $$\frac{\varphi \qquad \varphi \rightarrow \psi}{\psi}$$

E is the smallest classical modal logic.

EM is the logic **E** + *Mon*

PC 6. Propositional Calculus

E $\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$

Mon $\dfrac{\varphi \rightarrow \psi}{\Box\varphi \rightarrow \Box\psi}$

C $(\Box\varphi \wedge \Box\psi) \rightarrow \Box(\varphi \wedge \psi)$

N $\Box\top$

K $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$

RE $\dfrac{\varphi \leftrightarrow \psi}{\Box\varphi \leftrightarrow \Box\psi}$

Nec $\dfrac{\varphi}{\Box\varphi}$

MP $\dfrac{\varphi \quad \varphi \rightarrow \psi}{\psi}$

E is the smallest classical modal logic.

EM is the logic **E** + *Mon*

EC is the logic **E** + *C*

PC Propositional Calculus

E $\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$

Mon $\dfrac{\varphi \rightarrow \psi}{\Box\varphi \rightarrow \Box\psi}$

C $(\Box\varphi \wedge \Box\psi) \rightarrow \Box(\varphi \wedge \psi)$

N $\Box\top$

K $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$

RE $\dfrac{\varphi \leftrightarrow \psi}{\Box\varphi \leftrightarrow \Box\psi}$

Nec $\dfrac{\varphi}{\Box\varphi}$

MP $\dfrac{\varphi \quad \varphi \rightarrow \psi}{\psi}$

E is the smallest classical modal logic.

EM is the logic **E** + *Mon*

EC is the logic **E** + *C*

EMC is the smallest regular modal logic

PC Propositional Calculus

E $\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$

Mon $\dfrac{\varphi \rightarrow \psi}{\Box\varphi \rightarrow \Box\psi}$

C $(\Box\varphi \wedge \Box\psi) \rightarrow \Box(\varphi \wedge \psi)$

N $\Box\top$

K $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$

RE $\dfrac{\varphi \leftrightarrow \psi}{\Box\varphi \leftrightarrow \Box\psi}$

Nec $\dfrac{\varphi}{\Box\varphi}$

MP $\dfrac{\varphi \quad \varphi \rightarrow \psi}{\psi}$

E is the smallest classical modal logic.

EM is the logic **E** + *Mon*

EC is the logic **E** + *C*

EMC is the smallest regular modal logic

A logic is normal if it contains all instances of *E*, *C* and is closed under *Mon* and *Nec*

PC Propositional Calculus

E $\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$

Mon $\dfrac{\varphi \rightarrow \psi}{\Box\varphi \rightarrow \Box\psi}$

C $(\Box\varphi \wedge \Box\psi) \rightarrow \Box(\varphi \wedge \psi)$

N $\Box\top$

K $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$

RE $\dfrac{\varphi \leftrightarrow \psi}{\Box\varphi \leftrightarrow \Box\psi}$

Nec $\dfrac{\varphi}{\Box\varphi}$

MP $\dfrac{\varphi \quad \varphi \rightarrow \psi}{\psi}$

E is the smallest classical modal logic.

EM is the logic **E** + *Mon*

EC is the logic **E** + *C*

EMC is the smallest regular modal logic

K is the smallest normal modal logic

PC Propositional Calculus

E $\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$

Mon $\dfrac{\varphi \rightarrow \psi}{\Box\varphi \rightarrow \Box\psi}$

C $(\Box\varphi \wedge \Box\psi) \rightarrow \Box(\varphi \wedge \psi)$

N $\Box\top$

K $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$

RE $\dfrac{\varphi \leftrightarrow \psi}{\Box\varphi \leftrightarrow \Box\psi}$

Nec $\dfrac{\varphi}{\Box\varphi}$

MP $\dfrac{\varphi \quad \varphi \rightarrow \psi}{\psi}$

E is the smallest classical modal logic.

EM is the logic **E** + *Mon*

EC is the logic **E** + *C*

EMC is the smallest regular modal logic

K = **EMCN**

PC Propositional Calculus

E $\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$

Mon $\dfrac{\varphi \rightarrow \psi}{\Box\varphi \rightarrow \Box\psi}$

C $(\Box\varphi \wedge \Box\psi) \rightarrow \Box(\varphi \wedge \psi)$

N $\Box\top$

K $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$

RE $\dfrac{\varphi \leftrightarrow \psi}{\Box\varphi \leftrightarrow \Box\psi}$

Nec $\dfrac{\varphi}{\Box\varphi}$

MP $\dfrac{\varphi \quad \varphi \rightarrow \psi}{\psi}$

E is the smallest classical modal logic.

EM is the logic **E** $+$ *Mon*

EC is the logic **E** $+$ *C*

EMC is the smallest regular modal logic

K $= PC(+E) + K + Nec + MP$

Useful Fact

Theorem (Uniform Substitution)

The following rule can be derived in **E**

$$\frac{\psi \leftrightarrow \psi'}{\varphi \leftrightarrow \varphi[\psi/\psi']}$$

Some Facts (1)

Theorem

The logic **E** *is sound and strongly complete with respect to the class of all neighborhood frames.*

Some Facts (1)

Theorem
The logic **E** *is sound and strongly complete with respect to the class of all neighborhood frames.*

Lemma
If $C \in \mathbf{L}$, *then* $\langle M_{\mathbf{L}}, N_{\mathbf{L}}^{min} \rangle$ *is closed under finite intersections.*

Theorem
The logic **EC** *is sound and strongly complete with respect to the class of neighborhood frames that are closed under intersections.*

Some Facts (2)

Fact: The canonical model for **EM** is not closed under supersets.

Some Facts (2)

Fact: The canonical model for **EM** is not closed under supersets.

Lemma

Suppose that $\mathbb{M} = sup(\mathbb{M}^{min}_{\mathbf{EM}})$*. Then* $\mathbb{M}$ *is canonical for* **EM**.

Theorem

The logic **EM** *is sound and strongly complete with respect to the class of supplemented frames.*

Some Facts (2)

Theorem

The logic **K** *is sound and strongly complete with respect to the class of filters.*

Theorem

The logic **K** *is sound and strongly complete with respect to the class of augmented frames.*

Incompleteness?

Are all modal logics complete with respect to some class of neighborhood frames?

Incompleteness?

Are all modal logics complete with respect to some class of neighborhood frames? No

Incompleteness

Martin Gerson. *The Inadequacy of Neighbourhood Semantics for Modal Logic*. Journal of Symbolic Logic (1975).

Presents two logics $\mathbf{L}$ and $\mathbf{L}'$ that are incomplete with respect to neighborhood semantics.

Incompleteness

Martin Gerson. *The Inadequacy of Neighbourhood Semantics for Modal Logic*. Journal of Symbolic Logic (1975).

Presents two logics **L** and **L**′ that are incomplete with respect to neighborhood semantics.

(there are formulas φ and φ' that are valid in the class of frames for **L** and **L**′ respectively, but φ and φ' are not deducible in the respective logics).

Incompleteness

Martin Gerson. *The Inadequacy of Neighbourhood Semantics for Modal Logic*. Journal of Symbolic Logic (1975).

Presents two logics $\mathbf{L}$ and $\mathbf{L}'$ that are incomplete with respect to neighborhood semantics.

$\mathbf{L}$ is between $\mathbf{T}$ and $\mathbf{S4}$

$\mathbf{L}'$ is above $\mathbf{S4}$ (adapts Fine's incomplete logic)

Comparing Relational and Neighborhood Semantics

Comparing Relational and Neighborhood Semantics

Fact: If a (normal) modal logic is complete with respect to some class of relational frames then it is complete with respect to some class of neighborhood frames.

What about the converse?

Are there normal modal logics that are incomplete with respect to relational semantics, but complete with respect to neighborhood semantics?

Comparing Relational and Neighborhood Semantics

Fact: If a (normal) modal logic is complete with respect to some class of relational frames then it is complete with respect to some class of neighborhood frames.

What about the converse?

Are there normal modal logics that are incomplete with respect to relational semantics, but complete with respect to neighborhood semantics? Yes!

Comparing Relational and Neighborhood Semantics

There is

- an extension of **K**

D. Gabbay. *A normal logic that is complete for neighborhood frames but not for Kripke frames*. Theoria (1975).

Comparing Relational and Neighborhood Semantics

There is

- an extension of **K**

D. Gabbay. *A normal logic that is complete for neighborhood frames but not for Kripke frames*. Theoria (1975).

- An extension of **T**

M. Gerson. *A Neighbourhood frame for T with no equivalent relational frame*. Zeitschr. J. Math. Logik und Grundlagen (1976).

Comparing Relational and Neighborhood Semantics

There is

- an extension of **K**

D. Gabbay. *A normal logic that is complete for neighborhood frames but not for Kripke frames*. Theoria (1975).

- An extension of **T**

M. Gerson. *A Neighbourhood frame for T with no equivalent relational frame*. Zeitschr. J. Math. Logik und Grundlagen (1976).

- An extension of **S4**

M. Gerson. *An Extension of S4 Complete for the Neighbourhood Semantics but Incomplete for the Relational Semantics*. Studia Logica (1975).

✓ Basic Modal Logic

✓ Weak Systems of Modal Logic

⇒ Combining Logics

⇒ Comparing Logics

Literature

A. Kurucz. *Combining modal logics*. in: P. Blackburn, J. van Benthem and F. Wolter (eds.) Handbook of Modal Logic, Studies in Logic and Practical Reasoning 3, Elsevier, 2007 pp. 869 - 924.

D. Gabbay, A. Kurucz, F. Wolter and M. Zakharyaschev. *Many-Dimensional Modal Logics: Theory and Applications*. Studies in Logic and the Foundations of Mathematics 148, Elsevier, 2003.

Transfer Results

Given a family $\mathbf{L}$ of modal logics and a *combination method* C, do certain properties of the component logics $\mathbf{L} \in \mathsf{L}$ transfer to their *combination* C(L)?

Transfer Results

Given a family $\mathbf{L}$ of modal logics and a *combination method* C, do certain properties of the component logics $\mathbf{L} \in \mathsf{L}$ transfer to their *combination* C(L)?

Typical Assumptions:

1. C is only defined on *finite* families $\mathbf{L}$ of modal logics
2. C(L) is a (multi-)modal logic
3. C(L) is an extension of each component logic $\mathbf{L} \in \mathsf{L}$

Transfer Results

Given a family **L** of modal logics and a *combination method* C, do certain properties of the component logics $\mathbf{L} \in$ L transfer to their *combination* C(L)?

Typical Assumptions:

1. C is only defined on *finite* families **L** of modal logics
2. C(L) is a (multi-)modal logic
3. C(L) is an extension of each component logic $\mathbf{L} \in$ L

Does (recursive) axiomatizability, decidability, complexity transfer?

Fusions

Let $\mathbf{L}_1$ and $\mathbf{L}_2$ be two normal (multi-)modal logics

Fusions

Let $\mathbf{L}_1$ and $\mathbf{L}_2$ be two normal (multi-)modal logics with disjoint sets of modal operators (say $\{\Box_1, \ldots, \Box_n\}$ and $\{\Box_{n+1}, \ldots, \Box_{n+m}\}$).

Fusions

Let $\mathbf{L}_1$ and $\mathbf{L}_2$ be two normal (multi-)modal logics with disjoint sets of modal operators (say $\{\Box_1, \ldots, \Box_n\}$ and $\{\Box_{n+1}, \ldots, \Box_{n+m}\}$).

The **fusion** of $\mathbf{L}_1$ and $\mathbf{L}_2$, denoted $\mathbf{L}_1 \oplus \mathbf{L}_2$, is the smallest normal modal logic in the joint language containing both $\mathbf{L}_1$ and $\mathbf{L}_2$.

If $\mathcal{C}_1 = \{\langle W, R_1, \ldots, R_n\rangle\}$ and $\mathcal{C}_2 = \{\langle W, S_1, \ldots, S_m\rangle\}$ are classes of Kripke frames, let $\mathcal{C}_1 \oplus \mathcal{C}_2 = \{\langle W, R_1, \ldots, R_n, S_1, \ldots, S_m\rangle\}$

Fusions

Let $\mathbf{L}_1$ and $\mathbf{L}_2$ be two normal (multi-)modal logics with disjoint sets of modal operators (say $\{\Box_1, \ldots, \Box_n\}$ and $\{\Box_{n+1}, \ldots, \Box_{n+m}\}$).

The **fusion** of $\mathbf{L}_1$ and $\mathbf{L}_2$, denoted $\mathbf{L}_1 \oplus \mathbf{L}_2$, is the smallest normal modal logic in the joint language containing both $\mathbf{L}_1$ and $\mathbf{L}_2$.

If $\mathcal{C}_1 = \{\langle W, R_1, \ldots, R_n \rangle\}$ and $\mathcal{C}_2 = \{\langle W, S_1, \ldots, S_m \rangle\}$ are classes of Kripke frames, let $\mathcal{C}_1 \oplus \mathcal{C}_2 = \{\langle W, R_1, \ldots, R_n, S_1, \ldots, S_m \rangle\}$

Theorem. $\mathbf{L}_1 \oplus \mathbf{L}_2$ is a conservative extension of (consistent) $\mathbf{L}_1$ and $\mathbf{L}_2$.

Fusions

Let $\mathbf{L}_1$ and $\mathbf{L}_2$ be two normal (multi-)modal logics with disjoint sets of modal operators (say $\{\Box_1, \ldots, \Box_n\}$ and $\{\Box_{n+1}, \ldots, \Box_{n+m}\}$).

The **fusion** of $\mathbf{L}_1$ and $\mathbf{L}_2$, denoted $\mathbf{L}_1 \oplus \mathbf{L}_2$, is the smallest normal modal logic in the joint language containing both $\mathbf{L}_1$ and $\mathbf{L}_2$.

If $\mathcal{C}_1 = \{\langle W, R_1, \ldots, R_n \rangle\}$ and $\mathcal{C}_2 = \{\langle W, S_1, \ldots, S_m \rangle\}$ are classes of Kripke frames, let $\mathcal{C}_1 \oplus \mathcal{C}_2 = \{\langle W, R_1, \ldots, R_n, S_1, \ldots, S_m \rangle\}$

Theorems. $\mathbf{L}_1 \oplus \mathbf{L}_2$ is a conservative extension of (consistent) $\mathbf{L}_1$ and $\mathbf{L}_2$. If $\mathbf{L}_1$ and $\mathbf{L}_2$ are *characterized* by $\mathcal{C}_1$ and $\mathcal{C}_2$ respectively, then $\mathbf{L}_1 \oplus \mathbf{L}_2$ is characterized by $\mathcal{C}_1 \oplus \mathcal{C}_2$.

Fusions

Let $\mathbf{L}_1$ and $\mathbf{L}_2$ be two normal (multi-)modal logics with disjoint sets of modal operators (say $\{\Box_1, \ldots, \Box_n\}$ and $\{\Box_{n+1}, \ldots, \Box_{n+m}\}$).

The **fusion** of $\mathbf{L}_1$ and $\mathbf{L}_2$, denoted $\mathbf{L}_1 \oplus \mathbf{L}_2$, is the smallest normal modal logic in the joint language containing both $\mathbf{L}_1$ and $\mathbf{L}_2$.

If $\mathcal{C}_1 = \{\langle W, R_1, \ldots, R_n\rangle\}$ and $\mathcal{C}_2 = \{\langle W, S_1, \ldots, S_m\rangle\}$ are classes of Kripke frames, let $\mathcal{C}_1 \oplus \mathcal{C}_2 = \{\langle W, R_1, \ldots, R_n, S_1, \ldots, S_m\rangle\}$

Theorems. Many other properties transfer (eg., decidability). See the Kurucz paper for details.

Products

Suppose $\mathcal{F}_1 = \langle W_1, R_1^1, \ldots, R_1^1 \rangle$ and $\mathcal{F}_2 = \langle W_2, R_2^1, \ldots, R_2^m \rangle$ be Kripke frames. Define the **product**:

$$\mathcal{F}_1 \times \mathcal{F}_2 = \langle W_1 \times W_2, R_h^1, \ldots, R_h^n, R_v^1, \ldots, R_v^m \rangle$$

where

$$\begin{array}{lll} (w_1, w_2) R_h^i (v_1, v_2) & \text{iff} & w_1 R_1^i v_1 \text{ and } w_2 = v_2 \\ (w_1, w_2) R_v^i (v_1, v_2) & \text{iff} & w_2 R_2^i v_2 \text{ and } w_1 = v_1 \end{array}$$

Let $\mathbf{L}_1$, $\mathbf{L}_2$ be (Kripke complete) logics, the **product** is

$$\mathbf{L}_1 \times \mathbf{L}_2 = \mathrm{Log}(\{\mathcal{F}_1 \times \mathcal{F}_2 \mid \mathcal{F}_1 \in \mathrm{Fr}(\mathbf{L}_1) \text{ and } \mathcal{F}_2 \in \mathrm{Fr}(\mathbf{L}_2)\})$$

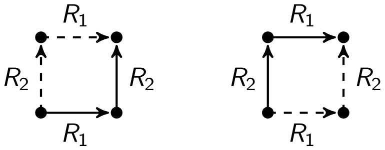
Products

Let $\mathbf{L}_1$ and $\mathbf{L}_2$ be Kripke complete (uni-)modal logics. Let $[\mathbf{L}_1, \mathbf{L}_2]$ be the smallest modal logic containing $\mathbf{L}_1, \mathbf{L}_2$ and the axioms:

Products

Let $\mathbf{L}_1$ and $\mathbf{L}_2$ be Kripke complete (uni-)modal logics. Let $[\mathbf{L}_1, \mathbf{L}_2]$ be the smallest modal logic containing $\mathbf{L}_1, \mathbf{L}_2$ and the axioms:

(comm) $\Diamond_1 \Diamond_2 p \leftrightarrow \Diamond_2 \Diamond_1 p$

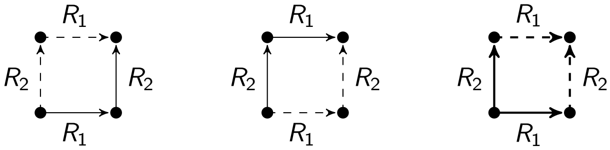


Products

Let $\mathbf{L}_1$ and $\mathbf{L}_2$ be Kripke complete (uni-)modal logics. Let $[\mathbf{L}_1, \mathbf{L}_2]$ be the smallest modal logic containing $\mathbf{L}_1, \mathbf{L}_2$ and the axioms:

$$\text{(comm)} \quad \Diamond_1 \Diamond_2 p \leftrightarrow \Diamond_2 \Diamond_1 p$$

$$\text{(cr)} \quad \Diamond_1 \Box_2 p \rightarrow \Box_2 \Diamond_1 p$$



Products: Some Interesting Facts

Products: Some Interesting Facts

- $[\mathbf{L}_1, \ldots, \mathbf{L}_n] \subseteq L_1 \times \cdots \times L_n$

Products: Some Interesting Facts

- $[\mathbf{L}_1, \ldots, \mathbf{L}_n] \subseteq L_1 \times \cdots \times L_n$
- very few transfer results: eg., $[\mathbf{K4}, \mathbf{K4}] = \mathbf{K4} \times \mathbf{K4}$, but is *undecidable*.

Products: Some Interesting Facts

- $[\mathbf{L}_1, \ldots, \mathbf{L}_n] \subseteq L_1 \times \cdots \times L_n$
- very few transfer results: eg., $[\mathbf{K4}, \mathbf{K4}] = \mathbf{K4} \times \mathbf{K4}$, but is *undecidable*.
- $[\mathbf{S4}, \mathsf{Log}(\langle \omega, < \rangle)]$ is Π^1_1-hard (another example coming later...)

Products: Some Interesting Facts

- $[\mathbf{L}_1, \ldots, \mathbf{L}_n] \subseteq L_1 \times \cdots \times L_n$
- very few transfer results: eg., $[\mathbf{K4}, \mathbf{K4}] = \mathbf{K4} \times \mathbf{K4}$, but is *undecidable*.
- $[\mathbf{S4}, \mathrm{Log}(\langle \omega, < \rangle)]$ is Π^1_1-hard (another example coming later...)

The general situation is very interesting, but beyond the scope of this course (see the Kurucz paper).

Products: Some Interesting Facts

- $[\mathbf{L}_1, \ldots, \mathbf{L}_n] \subseteq L_1 \times \cdots \times L_n$
- very few transfer results: eg., $[\mathbf{K4}, \mathbf{K4}] = \mathbf{K4} \times \mathbf{K4}$, but is *undecidable*.
- $[\mathbf{S4}, \mathrm{Log}(\langle \omega, < \rangle)]$ is Π^1_1-hard (another example coming later...)

The general situation is very interesting, but beyond the scope of this course (see the Kurucz paper).

We will see many examples of combining/merging modal logics during the course...

Background

✓ Basic Modal Logic

✓ Weak Systems of Modal Logic

✓ Combining Logics

⇒ Comparing Logics

What is the precise relationship between Neighborhood Models and Relational Models?

What is the precise relationship between Neighborhood Models and Relational Models?

We can *simulate* any non-normal modal logic with a bi-modal normal modal logic.

The key idea is to replace neighborhood models with a two-sorted Kripke model.

The key idea is to replace neighborhood models with a two-sorted Kripke model.

Given a neighborhood model $\mathcal{M} = \langle W, N, V \rangle$, define a Kripke model $\mathcal{M}^\circ = \langle W', R_N, R_{\not\ni}, R_N, Pt, V \rangle$ as follows:

- $W' = W \cup \wp(W)$

The key idea is to replace neighborhood models with a two-sorted Kripke model.

Given a neighborhood model $\mathcal{M} = \langle W, N, V\rangle$, define a Kripke model $\mathcal{M}^\circ = \langle W', R_N, R_{\not\ni}, R_N, Pt, V\rangle$ as follows:

- $W' = W \cup \wp(W)$
- $R_\ni = \{(u, w) \mid w \in W, u \in \wp(W), w \in u\}$

The key idea is to replace neighborhood models with a two-sorted Kripke model.

Given a neighborhood model $\mathcal{M} = \langle W, N, V \rangle$, define a Kripke model $\mathcal{M}^\circ = \langle W', R_N, R_{\not\ni}, R_N, Pt, V \rangle$ as follows:

- $W' = W \cup \wp(W)$
- $R_{\ni} = \{(u, w) \mid w \in W, u \in \wp(W), w \in u\}$
- $R_{\not\ni} = \{(u, w) \mid w \in W, u \in \wp(W), w \notin u\}$

The key idea is to replace neighborhood models with a two-sorted Kripke model.

Given a neighborhood model $\mathcal{M} = \langle W, N, V \rangle$, define a Kripke model $\mathcal{M}^\circ = \langle W', R_N, R_{\not\ni}, R_N, Pt, V \rangle$ as follows:

- $W' = W \cup \wp(W)$
- $R_\ni = \{(u, w) \mid w \in W, u \in \wp(W), w \in u\}$
- $R_{\not\ni} = \{(u, w) \mid w \in W, u \in \wp(W), w \notin u\}$
- $R_N = \{(w, u) \mid w \in W, u \in \wp(W), u \in N(w)\}$

The key idea is to replace neighborhood models with a two-sorted Kripke model.

Given a neighborhood model $\mathcal{M} = \langle W, N, V \rangle$, define a Kripke model $\mathcal{M}^\circ = \langle W', R_N, R_{\not\ni}, R_N, Pt, V \rangle$ as follows:

- $W' = W \cup \wp(W)$
- $R_\ni = \{(u, w) \mid w \in W, u \in \wp(W), w \in u\}$
- $R_{\not\ni} = \{(u, w) \mid w \in W, u \in \wp(W), w \notin u\}$
- $R_N = \{(w, u) \mid w \in W, u \in \wp(W), u \in N(w)\}$
- $Pt = W$

The key idea is to replace neighborhood models with a two-sorted Kripke model.

Given a neighborhood model $\mathcal{M} = \langle W, N, V \rangle$, define a Kripke model $\mathcal{M}^\circ = \langle W', R_N, R_{\not\ni}, R_N, Pt, V \rangle$ as follows:

- $W' = W \cup \wp(W)$
- $R_\ni = \{(u, w) \mid w \in W, u \in \wp(W), w \in u\}$
- $R_{\not\ni} = \{(u, w) \mid w \in W, u \in \wp(W), w \not\in u\}$
- $R_N = \{(w, u) \mid w \in W, u \in \wp(W), u \in N(w)\}$
- $Pt = W$

Let $\mathcal{L}'$ be the language

$$\varphi := p \mid \neg\varphi \mid \varphi \wedge \psi \mid [\ni]\varphi \mid [\not\ni]\varphi \mid [N]\varphi \mid \mathsf{Pt}$$

where $p \in \mathsf{At}$ and Pt is a unary modal operator.

Define $ST : \mathcal{L} \rightarrow \mathcal{L}'$ as follows

Define $ST : \mathcal{L} \to \mathcal{L}'$ as follows

- $ST(p) = p$

Define $ST : \mathcal{L} \rightarrow \mathcal{L}'$ as follows

- $ST(p) = p$
- $ST(\neg\varphi) = \neg ST(\varphi)$

Define $ST : \mathcal{L} \to \mathcal{L}'$ as follows

- $ST(p) = p$
- $ST(\neg\varphi) = \neg ST(\varphi)$
- $ST(\varphi \wedge \psi) = ST(\varphi) \wedge ST(\varphi)$

Define $ST : \mathcal{L} \rightarrow \mathcal{L}'$ as follows

- $ST(p) = p$
- $ST(\neg\varphi) = \neg ST(\varphi)$
- $ST(\varphi \wedge \psi) = ST(\varphi) \wedge ST(\varphi)$
- $ST(\Box\varphi) = \langle N \rangle([\ni]ST(\varphi) \wedge [\not\ni]\neg ST(\varphi))$

Define $ST : \mathcal{L} \rightarrow \mathcal{L}'$ as follows

- $ST(p) = p$
- $ST(\neg\varphi) = \neg ST(\varphi)$
- $ST(\varphi \wedge \psi) = ST(\varphi) \wedge ST(\varphi)$
- $ST(\Box\varphi) = \langle N \rangle([\ni]ST(\varphi) \wedge [\not\ni]\neg ST(\varphi))$

Lemma

For each neighborhood model $\mathcal{M} = \langle W, N, V \rangle$ and each formula $\varphi \in \mathcal{L}$, for any $w \in W$,

$$\mathcal{M}, w \models \varphi \text{ iff } \mathcal{M}^\circ, w \models ST(\varphi)$$

Monotonic Models

Lemma

On Monotonic Models $\langle N\rangle([\ni]ST(\varphi) \wedge [\not\ni]\neg ST(\varphi))$ *is equivalent to* $\langle N\rangle([\ni]ST(\varphi))$

What can we infer from the fact that bi-modal normal modal logic can simulate non-normal modal logics?

What can we infer from the fact that bi-modal normal modal logic can simulate non-normal modal logics?

Can we read off a notion of bisimulation?

What can we infer from the fact that bi-modal normal modal logic can simulate non-normal modal logics?

Can we read off a notion of bisimulation? Not clear.

H. Hvid Hansen, C. Kupke and EP. *Neighbourhood Structures: Bisimilarity and Basic Model Theory*. Logical Methods in Computer Science, 5:2, 2009.

What can we infer from the fact that bi-modal normal modal logic can simulate non-normal modal logics?

Can we read off a notion of bisimulation? Not clear.

H. Hvid Hansen, C. Kupke and EP. *Neighbourhood Structures: Bisimilarity and Basic Model Theory*. Logical Methods in Computer Science, 5:2, 2009.

- Decidability of the satisfiability problem
- Canonicity
- Salqhvist Theorem

O. Gasquet and A. Herzig. *From Classical to Normal Modal Logic*. in *Proof Theory of Modal Logic*, 1996, pgs. 293 - 311.

M. Kracht and F. Wolter. *Normal Monomodal Logics can Simulate all Others*. Journal of Symbolic Logic, Volume 64, Issue 1 (1999), 99-138.

H. Hansen (Chapter 10). *Monotonic Modal Logics*. 2003.

Background

- ✓ Basic Modal Logic
- ✓ Weak Systems of Modal Logic
- ✓ Combining Logics
- ✓ Comparing Logics

Ingredients of a Logical Analysis of Rational Agency

Ingredients of a Logical Analysis of Rational Agency

- What are the basic building blocks?

Ingredients of a Logical Analysis of Rational Agency

- What are the basic building blocks? (the nature of time (continuous or discrete/branching or linear), how (primitive) *events* or *actions* are represented, how *causal* relationships are represented and what constitutes a *state of affairs*.)

Ingredients of a Logical Analysis of Rational Agency

- What are the basic building blocks? (the nature of time (continuous or discrete/branching or linear), how (primitive) *events* or *actions* are represented, how *causal* relationships are represented and what constitutes a *state of affairs*.)

- Single agent vs. many agents.

Ingredients of a Logical Analysis of Rational Agency

- What are the basic building blocks? (the nature of time (continuous or discrete/branching or linear), how (primitive) *events* or *actions* are represented, how *causal* relationships are represented and what constitutes a *state of affairs*.)
- Single agent vs. many agents.
- What the the primitive operators?
  - Informational attitudes
  - Motivational attitudes
  - Normative attitudes

Ingredients of a Logical Analysis of Rational Agency

- What are the basic building blocks? (the nature of time (continuous or discrete/branching or linear), how (primitive) *events* or *actions* are represented, how *causal* relationships are represented and what constitutes a *state of affairs*.)

- Single agent vs. many agents.

- What the the primitive operators?
  - Informational attitudes
  - Motivational attitudes
  - Normative attitudes

- Static vs. dynamic

Ingredients of a Logical Analysis of Rational Agency

- ⇒ informational attitudes (eg., knowledge, belief, certainty)
- ⇒ time, actions and ability
- ⇒ motivational attitudes (eg., preferences)
- ⇒ group notions (eg., common knowledge and coalitional ability)
- ⇒ normative attitudes (eg., obligations)

End of Part 1.