Chapter 2

Core Theory

The previous chapter established that neighborhood structures with the basic propositional modal language is an interesting and well-motivated logical framework. In this chapter, I move away from questions of motivation to explore the logical theory of neighborhoods structures. The questions that we will discuss fall into two broad categories. The first category includes the standard logical questions about axiomatic completeness (Section 2.2), decidability of the satisfiability problem (Section 2.3), and model-theoretic questions (Section 2.5). The second category of questions focus on how neighborhood semantics should be viewed within the broader logical landscape. Section 2.1 explores the relationship between neighborhood semantics and other mathematical structures that have been used as a semantics for the basic modal language. Section 2.4 focuses on the precise relationship between neighborhood models and relational models.

2.1 Neighborhoods Semantics in the Broader Logical Landscape

This book is focused on neighborhood semantics for modal logic. Of course, there are many other types of models that can be used as a semantics for the basic modal language. It is important to understand where neighborhood models fit within the broader space of semantics for the basic modal language.

My goal in this section is to show that certain classes of neighborhood models are *equivalent* to other classes models which are used as a semantics for the basic modal language. In order to express this more formally, I need some notation. Suppose that M is a class of models (either neighborhood models or some other class of models). For a model $\mathcal{M} \in \mathsf{M}$, let $dom(\mathcal{M})$ denote the **domain** of $\mathcal{M}$, i.e., the set of states, or possible worlds, in $\mathcal{M}$. A pair $\mathcal{M}, w$ where $w \in dom(\mathcal{M})$ is called a **pointed model**. Each class of models M is associated with a comes with a notion of *truth* for the basic modal language $\mathcal{L}(\mathsf{At})$. Formally, it is a relation, denoted $\models_{\mathsf{M}}$, between pointed models and formulas $\varphi \in \mathcal{L}(\mathsf{At})$. I can now state the notion of equivalence between models that will be used in this Section.

Definition 2.1 (Modal Equivalence) Suppose that $\mathcal{L}$ is a modal language, and M and M' are two classes of models for $\mathcal{L}$. Let $\mathcal{M}, w$ be a pointed models from M and $\mathcal{M}', w'$ be a pointed model from M'. Say that $\mathcal{M}, w$ is $\mathcal{L}$-equivalent to $\mathcal{M}', w'$, denoted $\mathcal{M}, w \equiv_{\mathcal{L}} \mathcal{M}', w'$, provided $Th_{\mathcal{L}}(\mathcal{M}, w) = Th_{\mathcal{L}}(\mathcal{M}', w')$. If $\mathcal{L}$ is the basic modal language, then we say $\mathcal{M}, w$ and $\mathcal{M}, w'$ are **modally equivalent**. $\lhd$

The above definition can be lifted to classes of models as follows: A class of models M is $\mathcal{L}$-equivalent to a class of models M' provided for each pointed model $\mathcal{M}, w$ from M, there exists a pointed model $\mathcal{M}', w'$ from M' such that $\mathcal{M}, w \equiv_{\mathcal{L}} \mathcal{M}', w'$, and *vice versa*. Typically, demonstrating that M and M' are modally equivalent involves showing how to transform models from M into models from M' and, conversely, how to transform models from M' into models from M. After introducing the different classes of models, I will prove the following results:

- The class $\mathsf{K} = \{\mathcal{M} \mid \mathcal{M} \text{ is a relational model }\}$ is modally equivalent to the class $\mathsf{M}_{aug} = \{\mathfrak{M} \mid \mathfrak{M} \text{ is an augmented neighborhood model}\}$
- The class $\mathsf{K}^n = \{\mathcal{M}^n \mid \mathcal{M}^n \text{ is an } n\text{-ary relational model }\}$ is modally equivalent to the class $\mathsf{M}_{reg} = \{\mathfrak{M} \mid \mathfrak{M} \text{ is a non-trivial regular neighborhood model}\}$
- The class $\mathsf{T} = \{\mathcal{M}^T \mid \mathcal{M}^T \text{ is a topological model }\}$ is modally equivalent to the class $\mathsf{M}_{\mathsf{S4}} = \{\mathfrak{M} \mid \mathfrak{M} \text{ is an } \mathbf{S4} \text{ neighborhood model}\}$

The relationship with *plausibility models* is more subtle. This will be discussed in Section 2.1.4.

2.1.1 Relational Models

Let R be a relation on a non-empty set W (i.e., $R \subseteq W \times W$). There are two natural functions associated with R:

1. $R^{\rightarrow} : W \rightarrow \wp(W)$: for each $w \in W$, let $R^{\rightarrow}(w) = \{v \mid wRv\}$.
2. $R^{\leftarrow} : \wp(W) \rightarrow \wp(W)$: for each $X \subseteq W$, $R^{\leftarrow}(X) = \{w \mid \exists v \in X \text{ such that } wRv\}$.

So, $R^{\rightarrow}$ maps each state w to the set of states that w can "see" via the relation R. The function $R^{\leftarrow}$ maps each subset $X \subseteq W$ to the set of states that can "see" some element of X (via the relation R).

Definition 2.2 (R-Necessity) Let R be a relation on a non-empty set W and $w \in W$. A set $X \subseteq W$ is **R-necessary at** w if $R^{\rightarrow}(w) \subseteq X$. Let $\mathcal{N}_w^R$ be the set of sets that are R-necessary at w (we simple write $\mathcal{N}_w$ if R is clear from context). That is, $\mathcal{N}_w^R = \{X \mid R^{\rightarrow}(w) \subseteq X\}$. $\lhd$

The following Lemma shows that the collection of necessary sets have very nice algebraic properties.

Lemma 2.3 *Let R be a relation on W. Then for each $w \in W$, $\mathcal{N}_w$ is augmented.*

Proof. Left as an exercise. QED

Furthermore, properties of R are reflected in these collection of sets.

Observation 2.4 *Let W be a set and $R \subseteq W \times W$.*

1. *If R is reflexive, then for each $w \in W$, $w \in \cap\mathcal{N}_w$*
2. *If R is transitive then for each $w \in W$, if $X \in \mathcal{N}_w$, then $\{v \mid X \in \mathcal{N}_v\} \in \mathcal{N}_w$.*

Proof. Suppose that R is reflexive. Let $w \in W$ be an arbitrary state. Suppose that $X \in \mathcal{N}_w$. Then since R is reflexive, wRw and hence $w \in R^{\rightarrow}(w)$. Therefore by the definition of $\mathcal{N}_w$, $w \in X$. Since X was an arbitrary element of $\mathcal{N}_w$, $w \in X$ for each $X \in \mathcal{N}_w$. Hence $w \in \cap\mathcal{N}_w$.

Suppose that R is transitive. Let $w \in W$ be an arbitrary state. Suppose that $X \in \mathcal{N}_w$. We must show $\{v \mid X \in \mathcal{N}_v\} \in \mathcal{N}_w$. That is, we must show $R^{\rightarrow}(w) \subseteq \{v \mid X \in \mathcal{N}_v\}$. Let $x \in R^{\rightarrow}(w)$. Then wRx. To complete the proof we need only show $X \in \mathcal{N}_x$. That is, we must show $R^{\rightarrow}(x) \subseteq X$. Since R is transitive, $R^{\rightarrow}(x) \subseteq R^{\rightarrow}(w)$ (why?). Hence since $R^{\rightarrow}(w) \subseteq X$, $R^{\rightarrow}(x) \subseteq X$. QED

Exercise 2.5 *State and prove analogous results for the situations when R is serial (for all $w \in W$, there exists a v such that wRv), Euclidean (for all $w, v, u \in W$, if wRv and wRu then vRu) and symmetric (for all $w, v \in W$, if wRv then vRw).*

The perspective that I want to stress in this Section is that relational frames and neighborhood frames are two different ways of presenting the same information. It should be clear that there is more freedom in defining which sets are necessary at a state using neighborhood structures. However, relational frames (cf. Definition A.2) are a simple and elegant semantics for the basic modal language. The question that interests us in this Section is: When is a neighborhood frame and a relational frame *equivalent*? As mentioned in the introduction, "equivalence" is defined in terms of modal equivalence.

Definition 2.6 Let W be a nonempty set of states, $\langle W, N\rangle$ a neighborhood frame and $\langle W, R\rangle$ be a relational frame. We say that $\langle W, N\rangle$ and $\langle W, R\rangle$ are **equivalent**[1] provided for all $X \subseteq W$, $X \in N(w)$ iff $X \in \mathcal{N}^R_w$.

It is a simple exercise to show that if a neighborhood frame and a relational frame are equivalent, then they are modal equivalent. The following two theorems show that augmented neighborhood frames are equivalent to relational frames.

Theorem 2.7 *Let $\langle W, R\rangle$ be a relational frame. Then there is an equivalent augmented neighborhood frame.*

[1] We study other, more general, notions of equivalence between structures later in Chapter **??**.

Proof. The proof is straightforward given Lemma 2.3: for each $w \in W$, let $N(w) = \mathcal{N}_w^R$ (where R is the relation under consideration). QED

Theorem 2.8 *Let $\langle W, N\rangle$ be an augmented neighborhood frame. Then there is an equivalent relational frame.*

Proof. Let $\langle W, N\rangle$ be a neighborhood frame. We must define a relation R_N on W. Since $\langle W, N\rangle$ is augmented, for each $w \in W$, $\cap N(w) \in N(w)$. For each $w, v \in W$, we say wR_Nv iff $v \in \cap N(w)$. To show $\langle W, R_N\rangle$ and $\langle W, N\rangle$ are equivalent, we must show for each $w \in W$, $\mathcal{N}_w = N(w)$. Let $w \in W$ and $X \subseteq W$. If $X \in \mathcal{N}_w^R$. Then $R_N^{\rightarrow}(w) \subseteq X$. Since $R_N^{\rightarrow}(w) = \cap N(w)$ and N contains its core, $R_N^{\rightarrow}(w) \in N(w)$. Furthermore, since N is supplemented and $R_N^{\rightarrow}(w) = \cap N(w) \subseteq X$, $X \in N(w)$. Suppose that $X \in N(w)$. Then clearly $\cap N(w) \subseteq X$. Hence $X \in \mathcal{N}_w$. QED

2.1.2 Generalized Relational Models

The models discussed in this section are intended to provide a natural semantics for so-called *non-adjunctive logics*. These are modal logics that do not include the axiom scheme C $((\Box\varphi \wedge \Box\psi) \rightarrow \Box(\varphi \wedge \psi))$. The following models were introduced by Schotch and Jennings in a series of papers (Schotch and Jennings, 1980; Jennings and Schotch, 1981, 1980) .

Definition 2.9 (n-ary Relational Model) An **n-ary relational model** (where $n \geq 2$) is a tuple $\langle W, R, V\rangle$ where W is a non-empty set and $R \subseteq W^n$ is an n-ary relation and $V : \mathsf{At} \rightarrow \wp(W)$ is a valuation function. ◁

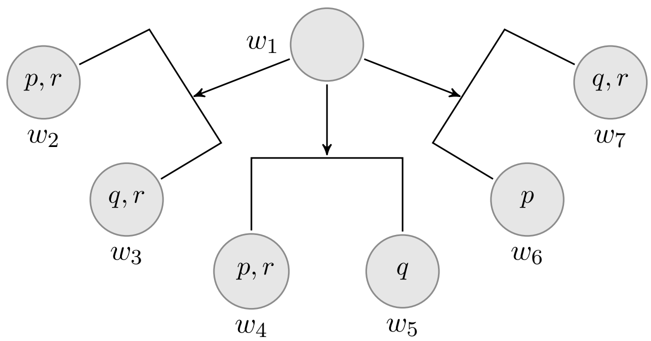
So, relational models for modal logic (cf. Definition A.2) are 2-ary models. The definition of truth for the basic modal language $\mathcal{L}(\mathsf{At})$ follows the usual pattern. Let $\mathcal{M}^n = \langle W, R, V\rangle$ be an n-ary relational model and $w \in W$. The Boolean connectives are defined as usual. The clauses for the modal operators are:

- $\mathcal{M}^n, w \models \Box\varphi$ iff for all $(w_1, \ldots, w_{n-1}) \in W^{n-1}$, if $(w, w_1, \ldots, w_{n-1}) \in R$, then there exists i such that $1 \leq i \leq n$ and $\mathcal{M}^n, w_i \models \varphi$.
- $\mathcal{M}^{n-1}, w \models \Diamond\varphi$ iff there exists $(w_1, \ldots, w_{n-1}) \in W^{n-1}$ such that $(w, w_1, \ldots, w_{n-1}) \in R$, and for all i such that $1 \leq i \leq n$, we have $\mathcal{M}^n, w_i \models \varphi$.

An n-ary frame is a pair $\langle W, R\rangle$ where $R \subseteq W^n$ is an n-ary relation. The usual logical notions of satisfiability and validity (with respect to models and frames) are defined as usual.

Example 2.10 (A 3-ary Relational Model) Let $\mathcal{M}^3 = \langle W, R, V\rangle$ be a 3-ary relational model for the modal language generated from the atomic propositions $\mathsf{At} = \{p, q, r\}$ where $W = \{w_1, w_2, w_3, w_4, w_5, w_6, w_7\}$; $R = \{(w_1, w_2, w_3), (w_1, w_4, w_5), (w_1, w_6, w_7)\}$; and

$V(p) = \{w_2, w_4, w_6\}$, $V(q) = \{w_3, w_5, w_7\}$ and $V(r) = \{w_2, w_3, w_7\}$. This model can be pictured as follows:



According to the above definition of truth for the modal operators on n-ary relational models, we have:

- $\mathcal{M}^3, w_1 \models \Box p$ (and $\mathcal{M}^3, w_1 \models \Box\neg p$);
- $\mathcal{M}^3, w_1 \models \Box q$ (and $\mathcal{M}^3, w_1 \models \Box\neg q$); and
- $\mathcal{M}^3, w_1 \not\models \Box(p \wedge q)$;

The above model shows that the axiom scheme C is not valid on the class of 3-ary relational models. Let $\mathfrak{C}^n$ be the class of all n-ary relational frames and $\mathfrak{C} = \bigcup_{n\geq 2} \mathfrak{C}^n$ the set of all n-ary relational frames. It is not hard to see that the monotonicity rule and necessitation are both valid on any class of n-ary frames. In fact, it has been shown that **EMN** is sound and complete for the class $\mathfrak{C}$ of all n-ary relational frames (Jennings and Schotch, 1981; Allen, 2005). Since $\mathfrak{C}^2$ is the class of all relational frames, C is valid on $\mathfrak{C}^2$.

Consider again the model $\mathcal{M}^3$ given in Example 2.10. Note that $\mathcal{M}^3, w_1 \models \Box r$, so we have $\mathcal{M}^3 \models \Box p \wedge \Box q \wedge \Box r$. While $\Box$ is not "closed under conjunction"[2], a weaker conjunctive closure condition is satisfied: $\mathcal{M}^3, w_1 \models \Box((p \wedge r) \vee (q \wedge r))$. The main interest in n-ary relational models is that they can be used to study a hierarchy of weaker and weaker conjunctive closure principles. For each $n \geq 2$, define the following formula:

$$(C^n) \qquad \bigwedge_{i=1}^{n} \Box\varphi_i \to \Box \bigvee_{1\leq k,l\leq n, k\neq l} (\varphi_k \wedge \varphi_l)$$

So, for example, C^3 is the formula

$$(\Box\varphi_1 \wedge \Box\varphi_2 \wedge \Box\varphi_3) \to \Box((\varphi_1 \wedge \varphi_2) \vee (\varphi_2 \wedge \varphi_3) \vee (\varphi_1 \wedge \varphi_3)).$$

Observation 2.11 *The formula C^3 is valid on any 3-ary relational frame.*

[2] In fact, we have $\mathcal{M}^3, w_1 \not\models \Box(p \wedge q)$, $\mathcal{M}^3, w_1 \not\models \Box(p \wedge r)$, and $\mathcal{M}^3, w_1 \not\models \Box(q \wedge r)$.

Proof. Suppose that $\mathcal{F}^3 \in \mathfrak{C}^3$ is a 3-ary relational frame with and $\mathcal{M}^3 = \langle W, R, V\rangle$ a model based on $\mathcal{F}^3$. Let $w \in W$ with $\mathcal{M}, w \models \Box\varphi_1 \wedge \Box\varphi_2 \wedge \Box\varphi_3$. We must show that $\mathcal{M}^3, w \models \Box((\varphi_1 \wedge \varphi_2) \vee (\varphi_2 \wedge \varphi_3) \vee (\varphi_1 \wedge \varphi_3))$. Suppose that (w, w_1, w_2) is any triple of states from W such that $(w, w_1, w_2) \in R$. Then, by assumption, for each $k \in \{1, 2, 3\}$ there exists $j_k \in \{1, 2\}$ such that $\mathcal{M}^3, w_{j_k} \models \varphi_k$. By the *pigeon hole principle*[3], there must be $k', k'' \in \{1, 2, 3\}$ such that $j_{k'} = j_{k''}$. Thus, $\mathcal{M}^3, w_{j_{k'}} \models \varphi_{k'} \wedge \varphi_{k''}$. Since this is true for each triple (w, w_1, w_2) (though, in general, different pairs of formulas will be identified by the pigeon hole principle for each triple), we have $\mathcal{M}^3, w \models \Box((\varphi_1 \wedge \varphi_2) \vee (\varphi_2 \wedge \varphi_3) \vee (\varphi_1 \wedge \varphi_3))$, as desired. QED

Let $\mathbf{EMNC}^n$ be the smallest set of formulas containing all instances of D, all instances of C^n for each $n \geq 2$, and is closed under the monotonicity rule (RM) and necessitation (Nec). Then, as the reader is invited to check, we can show the following:

- For each $n \geq 2$, $C^n \vdash_{\mathbf{EMNC}^n} C^{n+1}$
- For each $n \geq 2$, C^n is valid on the class $\mathfrak{C}^n$ of n-ary relational frames, but not on the class $\mathfrak{C}^{n+1}$ of $(n+1)$-ary relational frames.

I conclude the discussion of n-ary relational frames by addressing two natural questions. The first question concerns the logic of n-ary relational frames for a fixed $n \geq 2$. Suppose that $\mathbf{L}(\mathfrak{C}^n) = \{\varphi \in \mathcal{L}(\mathsf{At}) \mid \text{for all } \mathcal{F}^n \in \mathfrak{C}^n, \mathcal{F}^n \models \varphi\}$. Schotch and Jennings (1981) have shown that:

$$\mathbf{EMN} = \bigcap_{n \geq 2} \mathbf{L}(\mathfrak{C}^n)$$

But a question remains: Are there natural axiomatizations of $\mathbf{L}(\mathfrak{C}^n)$ for each $n > 2$? A positive answer was provided by Apostoli and Brown (1995) (cf. also Nicholson et al., 2000):

Theorem 2.12 *(Apostoli and Brown, 1995; Nicholson et al., 2000) The logic* $\mathbf{EMNC}^n$ *is sound and complete for the class* $\mathfrak{C}^n$ *of n-ary relational frames.*

Consult the cited papers for proofs of this Theorem. The second question is: What exactly is the relationship with neighborhood structures? Martin Allen 2005) showed that every n-ary relational structure is modally equivalent to a finite monotonic neighborhood structure, and vice versa (cf. Arló-Costa (2005)).

Proposition 2.13 *Suppose that* $\mathfrak{M} = \langle W, N, V\rangle$ *is finite monotonic neighborhood model such that for all* $w \in W$, $N(w) \neq \emptyset$. *Then, there is an n-ary relational model* $\mathcal{M}^N = \langle W^N, R^N, V^N\rangle$ *that is modally equivalent to* $\mathfrak{M}$.

[3] The pigeon hole principle states that if m items are put into n containers, with $m > n$, then at least one container must contain more than one item. In this example, the containers are the worlds w_1 and w_2 and the items are the three formulas φ_1, φ_2 and φ_3.

Proof. Suppose that $\mathfrak{M} = \langle W, N, V\rangle$ is a finite monotonic neighborhood model. Recall that for each $w \in W$, $N^{nc}(w)$ is the non-monotonic core of $N(w)$ (see Definition 1.4). Let $m = \max\{|N^{nc}(w)| \mid w \in W\}$. Without loss of generality, we can assume that for all $w \in W$, there are exactly m sets in the non-monotonic core of $N(w)$. This may require modifying the neighborhood model $\mathfrak{M}$ by adding copies of states. Construct an n-ary relational model $\mathcal{M}^N = \langle W^N, R^N, V^N\rangle$ as follows: Let $W^N = W$, $V^N = V$ and define the n-ary relation R^N as follows. For each $w \in W$, let ν_w denote a sequence consisting of the m sets from $N^{nc}(w)$. For $1 \le k \le m$, let $\nu_{w,k}$ denote the kth set in the sequence. Then,

$$(w, w_1, \ldots, w_m) \in R^N \text{ iff } w_k \in \nu_{w,k}, \text{ for all } 1 \le k \le m.$$

To conclude the proof, we must show that $\mathfrak{M}$ and $\mathcal{M}^N$ are modally equivalent.

Claim 2.14 *For all $w \in W$, $\mathfrak{M}, w \equiv \mathcal{M}^N, w$.*

Proof of Claim 2.14. We must show that for all $\varphi \in \mathcal{L}(\mathsf{At})$, $[\![\varphi]\!]_{\mathfrak{M}} = [\![\varphi]\!]_{\mathcal{M}^N}$. The proof is by induction on the structure of formulas in $\mathcal{L}(\mathsf{At})$. The base case is $p \in \mathsf{At}$: $[\![p]\!]_{\mathfrak{M}} = V(p) = V^N(p) = [\![p]\!]_{\mathcal{M}^N}$. The proof for the Boolean connectives is standard. We only show the case of the modal operator.

Suppose that $w \in [\![\Box\varphi]\!]_{\mathfrak{M}}$. Then, $[\![\varphi]\!]_{\mathfrak{M}} \in N(w)$. This means that there is some $1 \le k \le m$ such that $\nu_{w,k} \subseteq [\![\varphi]\!]_{\mathfrak{M}}$. By construction, for all $(w, w_1, \ldots, w_m)$, if $(w, w_1, \ldots, w_m) \in R^N$, then $w_k \in \nu_{w,k} \subseteq [\![\varphi]\!]_{\mathfrak{M}} = [\![\varphi]\!]_{\mathcal{M}^N}$ (the latter equality follows from the induction hypothesis). Thus, $\mathcal{M}^N, w \models \Box\varphi$, i.e., $w \in [\![\Box\varphi]\!]_{\mathcal{M}^N}$. Hence, $[\![\Box\varphi]\!]_{\mathfrak{M}} \subseteq [\![\Box\varphi]\!]_{\mathcal{M}^N}$.

We must show $[\![\Box\varphi]\!]_{\mathcal{M}^N} \subseteq [\![\Box\varphi]\!]_{\mathfrak{M}}$. Suppose that $w \notin [\![\Box\varphi]\!]_{\mathfrak{M}}$. Then for all $1 \le k \le m$, $\nu_{w,k} \not\subseteq [\![\varphi]\!]_{\mathfrak{M}}$. Thus, for all $1 \le k \le m$, there exists a $v \notin [\![\varphi]\!]_{\mathfrak{M}}$ such that $v \in \nu_{w,k}$. By the induction hypothesis, $[\![\varphi]\!]_{\mathcal{M}^{\mathfrak{N}}} = [\![\varphi]\!]_{\mathfrak{M}}$. Thus, there exists a sequence $(v_1, \ldots, v_m) \in \Pi_{i=1}^m \nu_{w,i}$ such that for all $1 \le k \le m$, $v_k \notin [\![\varphi]\!]_{\mathfrak{M}} = [\![\varphi]\!]_{\mathcal{M}^N}$. For each $w \in W$, let $R^N(w) = \{(w_1, \ldots, w_m) \mid R^N(w, w_1, \ldots, w_m)\}$. By construction, $R^N(w) = \Pi_{i=1^m} \nu_{w,i}$. Hence, $(w, v_1, \ldots, v_m) \in R^N$; and so, $w \notin [\![\Box\varphi]\!]_{\mathcal{M}^N}$. QED

The following example illustrates the above construction.

Example 2.15 (Translating Neighborhood Models to n-ary Relational Modles) Suppose that $\mathfrak{M} = \langle W, N, V\rangle$ is a monotonic neighborhood model with $W = \{w, v\}$; $N(w) = \{\{w\}, \{v\}, \{w, v\}\}$ and $N(v) = \{\{w, v\}\}$; and $V(p) = \{w\}$ and $V(q) = \{v\}$. Note that $N^{nc}(w) = \{\{w\}, \{v\}\}$ and $N^{nc}(v) = \{\{w, v\}\}$. The fist step is to add copies of the states so that each neighborhood contains exactly two sets. To that end, let $\mathfrak{M}' = \langle W', N', V'\rangle$ with $W' = W \cup \{w', v'\}$, where w' is a copy of w and v' is a copy of v. So, $N'(v) = N'(v') = \{\{w, v\}, \{w', v'\}\}$ and $N'(w) = N'(w') = \{\{w\}, \{v\}\}$; and $V'(p) = \{w, w'\}$ and $V'(q) = \{v, v'\}$. The second step is to construct a 3-ary relational model $\mathcal{M}^{N'} = \langle W^{N'}, R^{N'}, V^{N'}\rangle$ where

- $W^{N'} = \{w, v, w', v'\}$;

- $R^{N'} = \{(w,w,v),(v,w,w'),(v,w,v'),(v,v,w'),(v,v,v')\}$; and
- $V^{N'} = V'$

The proof that every finite n-ary relational model can be transformed into a monotonic neighborhood model is more involved. Consult (Allen, 2005) for the details.

Proposition 2.16 *Suppose that $\mathcal{M}^n = \langle W, R, V\rangle$ is a finite n-ary relational model. Then, there is a finite, non-trivial, monotonic neighborhood model $\mathfrak{M}^R = \langle W^R, N^R, V^R\rangle$ that is modally equivalent to $\mathcal{M}^n$.*

I conclude this section by briefly mentioning two other semantics for non-normal modal logics that generalize relational semantics for modal logic.

Multi-Relational Models Lou Goble (2000) used models with a set of relations as a semantics for a deontic logic in which there are possible conflicting obligations arising from different normative sources (cf. also Governatori and Rotolo, 2005).

Definition 2.17 (Multi-Relational Model) Suppose that At is a set of atomic propositions. A **multi-relational model** is a triple $\langle W, \mathcal{R}, V\rangle$ where W is a non-empty set, $\mathcal{R} \subseteq \wp(W \times W)$ is a set of serial relations (i.e., for all $R \in \mathcal{R}$, for all $w \in W$, there exists $v \in W$ such that $w\ R\ v$) and $V : \mathsf{At} \to \wp(W)$ is a valuation function. $\triangleleft$

The definition of truth for the basic modal language $\mathcal{L}(\mathsf{At})$ follows the usual pattern. Let $\mathcal{M} = \langle W, \mathcal{R}, V\rangle$ be a multi-relational model and $w \in W$. The Boolean connectives are defined as usual. The clauses for the modal operators are:

- $\mathcal{M}, w \models \Box\varphi$ iff there exists $R \in \mathcal{R}$ such that for all $v \in W$, if $w\ R\ v$, then $\mathcal{M}, v \models \varphi$.
- $\mathcal{M}, w \models \Diamond\varphi$ iff for all $R \in \mathcal{R}$ there is a $v \in W$ such that $w\ R\ v$, then $\mathcal{M}, v \models \varphi$.

Note that according to the above definition, the relations in a multi-relational model $\langle W, \mathcal{R}, V\rangle$ are assumed to be serial. This means that in for all states $w \in W$, for all $R \in \mathcal{R}$, the set $R^{\to}(w) = \{v \mid w\ R\ v\}$ is non-empty. This assumption can be dropped, but doing so will lead to some complications. A state $w \in W$ is said to be a dead-end state with respect to a relation R provided $R^{\to}(w) = \emptyset$ (i.e., there are no states accessible from w). This means that if w is a dead-end state for a relation $R \in \mathcal{R}$ in a multi-relational models $\mathcal{M}$, then $\mathcal{M}, w \models \Box\bot$. When studying non-adjunctive logics, it is important to distinguish between situations in which $\Box\varphi \wedge \Box\neg\varphi$ is true and situations in which $\Box\bot$ is true. Ruling out dead-end states ensures that $\neg\Box\bot$ is valid.

Impossible Worlds Impossible worlds were first introduced into modal logic by Saul Kripke (1965) to provide a semantics for some historically important systems of modal logic weaker than **K**. Impossible worlds can be used in a variety of ways to weaken systems of modal logic (Berto, 2013). I will briefly discuss how to use impossible worlds to provide a semantics of regular modal logics. These are modal logics in which the necessitation rule is not valid (equivalently, modal logics that do not contain $\Box\top$ as an axiom).

Say that a world w is an **impossible world** if nothing is necessary (no formulas of the form $\Box\varphi$ are true at w) and everything is possible (all formulas of the form $\Diamond\varphi$ are true at W). The key idea is to distinguish between possible and impossible worlds in a relational model.

Definition 2.18 (Relational models with impossible worlds) A tuple $\langle W, W_N, R, V\rangle$ is a relational model with impossible worlds provided W is a non-empty set of worlds; $W_N \subsetneq W$ is a proper subset of W; $R \subseteq W \times W$; and $V : \mathsf{At} \to \wp(W)$. $\triangleleft$

Suppose that $\mathcal{M} = \langle W, W_N, R, V\rangle$ is a relational model with impossible worlds. Truth for the basic modal language is as usual except for the modal clause:

- $\mathcal{M}, w \models \Box\varphi$ iff $w \in W_N$ and for all $v \in W$, if $w\ R\ v$, then $\mathcal{M}, v \models \varphi$.

Adding impossible worlds to relational models is an elegant way to invalidate the necessitation rule (while keeping all other axioms and rules of normal modal logic intact). Consider any atomic proposition p. This, in turn, means that $\Box(p \vee \neg p)$ is valid on any model. However, it can be shown that $\Box\Box(p \vee \neg p)$ is not valid.

Suppose that $\mathcal{M} = \langle W, W_N, R, V\rangle$ is a relational model with impossible world consisting of worlds $W = \{w, v\}$ and $W_N = \{w\}$ with $R = \{(w, v)\}$. Since the interpretation of the Boolean connectives at both possible and impossible worlds, we have that $p\vee\neg p$ is valid on any relational model with impossible worlds (in particular, $\mathcal{M} \models p \vee \neg p$). However, since $v \notin W_V$, we have $\mathcal{M}, w \not\models \Box(p\vee\neg p)$. Thus, since wRv, we have $\mathcal{M}, w \not\models \Box\Box(p\vee\neg p)$.

There is much more to say about impossible worlds and how they can be used to model various non-normal modal logics. The interested reader is invited to consult (Priest, 2001) and (Berto, 2013), and references therein, for a more extensive discussion.

2.1.3 Topological Models

Much of the original motivation for neighborhood structures as a semantics form modal logics comes from elementary point-set topology. The idea is to think of the propositions in $N(w)$ to be *close* to the point w. In this section, I discuss the so-called topological semantics for modal logic. This semantics has been around for nearly 60 years and is usually attributed to McKinsey and Tarski McKinsey and Tarksi (1944). I start by reviewing some concepts from point-set topology. More information can be found in any point-set topology text book (Dugundji (1966) is an excellent choice).

Definition 2.19 (Topological Space) A **topological space** is a tuple $\langle W, \mathcal{T} \rangle$ where W is a nonempty set and

1. $W \in \mathcal{T}$ and $\emptyset \in W$;
2. $\mathcal{T}$ is closed under finite intersections; and
3. $\mathcal{T}$ is closed under arbitrary unions. $\triangleleft$

Elements $O \in \mathcal{T}$ are called **opens**. A set C such that $W - C \in \mathcal{T}$ is said to be **closed**. Given a topology $\langle W, \mathcal{T} \rangle$, let $\mathcal{T}_C$ be the collection of closed subsets of W, i.e., $\mathcal{T}_C = \{C \mid W - C \in \mathcal{T}\}$. The following observation is an easy consequence of the above definition.

Observation 2.20 *Let $\langle W, \mathcal{T} \rangle$ be a topological space. Then $\mathcal{T}_C$ has the following properties:*

1. $\emptyset, W \in \mathcal{T}_C$

2. $\mathcal{T}_C$ is closed under finite unions

3. $\mathcal{T}_C$ is closed under arbitrary intersections

Proof. The proof is left as an exercise for the reader. QED

Given a topological space $\langle W, \mathcal{T} \rangle$ and a point $w \in W$, a **neighborhood of** w is any open set that contains w. Let $\mathcal{T}_w = \{O \mid O \in \mathcal{T} \text{ and } w \in O\}$ be the collection of all neighborhoods of w.

Lemma 2.21 *Let $\langle W, \mathcal{T} \rangle$ be a topological space. Then for each $w \in W$, the collection $\mathcal{T}_w$ contains W, is closed under finite intersections and closed under arbitrary unions.*

Proof. The proof is left as an exercise for the reader. QED

Definition 2.22 (Neighborhood System) Let $\langle W, \mathcal{T} \rangle$ be a topological space. A pair $\langle W, N^{\mathcal{T}} \rangle$ is called a **neighborhood system** provided $N^{\mathcal{T}} : W \to \wp(\mathcal{T})$ is defined as follows: for all $w \in W$, $N(w) = \mathcal{T}_w$. $\triangleleft$

Let $\langle W, \mathcal{T} \rangle$ be a topological space and $X \subseteq W$ any set. The largest open subset of X is called the **interior** of X, denoted $Int(X)$. Formally,

$$Int(X) = \cup\{O \mid O \in \mathcal{T} \text{ and } O \subseteq X\}$$

The smallest closed set containing X is called the **closure** of X, denoted $Cl(X)$. Formally,

$$Cl(X) = \cap\{C \mid W - C \in \mathcal{T} \text{ and } X \subseteq C\}$$

It is easy to see that a set X is open if $Int(X) = X$ and closed if $Cl(X) = X$.

Lemma 2.23 *Let $\langle W, \mathcal{T}\rangle$ be a topological space and $X \subseteq W$. Then*

1. $Int(X \cap Y) = Int(X) \cap Int(Y)$
2. $Int(\emptyset) = \emptyset$, $Int(W) = W$
3. $Int(X) \subseteq X$
4. $Int(Int(X)) = Int(X)$
5. $Int(X) = W - Cl(W - X)$

Exercise 2.24 *Use the last fact in the above lemma to derive corresponding properties for the closure operator.*

I can now introduce topological models for the basic modal language.

Definition 2.25 (Topological Model) A **topological model** is a tuple $\mathcal{M}^T = \langle W, \mathcal{T}, V\rangle$, where $\langle W, \mathcal{T}\rangle$ is a topological space and $V : \mathsf{At} \to \wp(W)$ is a valuation function. ◁

Formulas of $\mathcal{L}(\mathsf{At})$ are interpreted at states $w \in W$. The boolean connectives and atomic propositions are interpreted as usual. I only give the definition of truth of the modal operator:

$$\mathcal{M}^T, w \models \Box\varphi \text{ iff } \text{ there is } O \in \mathcal{T}, \text{ such that } w \in O \text{ and for all } v \in O,\ \mathcal{M}^T, v \models \varphi$$

Notice the similarity between this definition an the definition of truth of the modal operator $\langle\ ]$. The only difference is the extra clause $w \in O$. However, I can restate the above clause using the function $w \mapsto \mathcal{T}_w$, where $\mathcal{T}_w$ is the set of neighborhoods of w. Then, the above clause can be written as

$$\mathcal{M}^T, w \models \Box\varphi \text{ iff } \text{ there is } O \in \mathcal{T}_w \text{ such that for all } v \in O,\ \mathcal{M}^T, v \models \varphi$$

Although this difference is a trivial change in terminology, it demonstrates the close connection between neighborhood frames and topological frames. Finally, it is easy to see that

$$[\![\Box\varphi]\!]_{\mathcal{M}^T} = Int([\![\varphi]\!]_{\mathcal{M}^T})$$

The seminal result of Tarski and McKinsey shows that **S4** (the smallest normal modal logic containing all instances of $\Box\varphi \to \varphi$ and $\Box\varphi \to \Box\Box\varphi$) is sound and complete for the class of all topologies. Topological semantics for modal logic is a very active area of research. A complete survey of the work in this area is beyond the scope of this book. The interested reader is invited to consult **??** for surveys. I conclude this brief introduction to topological models by describing the relationship with neighborhood models.

Definition 2.26 (S4 Neighborhood Frame) A neighborhood frame $\langle W, N\rangle$ is an **S4 neighborhood frame**[4] provided N satisfies the following properties. For each $w \in W$:

1. $N(w)$ is a consistent filter;
2. $w \in \cap N(w)$; and
3. for each $X \subseteq W$, if $X \in N(w)$, then $m_N(X) = \{v \mid X \in N(v)\} \in N(w)$. ◁

We can now show that topological models are equivalent to **S4** neighborhood models. In order to do this, I need some additional topological notions.

Definition 2.27 (Base for a Topology) Suppose that W is a non-empty set. A collection of sets $\mathcal{B} \subseteq \wp(W)$ is said to a be a **base** provided 1. $\bigcup \mathcal{B} = W$ and for all $X, Y \in \mathcal{B}$, if $x \in X \cap Y$, then there is a $Z \in \mathcal{B}$ such that $x \in Z \subseteq X \cap Y$. ◁

Bases are useful because they are a convenient way to define topological spaces:

Lemma 2.28 *Suppose that W is a non-empty set and $\mathcal{B}$ a base on W. Let $\mathcal{T}^{\mathcal{B}} = \{O \mid O = \bigcup \mathcal{X}$ for $\mathcal{X} \subseteq \mathcal{B}\}$. Then $\mathcal{T}^{\mathcal{B}}$ is a topology on W.*

Proof. Suppose that $\mathcal{B}$ is a base on W and $\mathcal{T}^{\mathcal{B}}$ is defined as above. We must show that $\mathcal{T}^{\mathcal{B}}$ satisfies the conditions in Definition 2.19. Since $\bigcup \mathcal{B} = W$, we have $W \in \mathcal{T}^{\mathcal{B}}$. Also, since $\cup \emptyset = \emptyset$, we have $\emptyset \in \mathcal{T}^{B}$. Clearly, $\mathcal{T}^{B}$ is closed under arbitrary unions: The union of sets from $\mathcal{T}^{\mathcal{B}}$ is again a set that is the union of sets from $\mathcal{B}$, so this set is in $\mathcal{T}^{\mathcal{B}}$. All that remains is to show that $\mathcal{T}^{\mathcal{B}}$ is closed under finite intersections. Given Lemma 1.10, it is enough to prove that $\mathcal{T}^{\mathcal{B}}$ is closed under binary intersections. Suppose that $X, Y \in \mathcal{T}^{\mathcal{B}}$. Then, there are $\mathcal{X} \subseteq \mathcal{B}$ and $\mathcal{Y} \subseteq \mathcal{B}$ such that $X = \bigcup \mathcal{X}$ and $Y = \bigcup \mathcal{Y}$. Now, a simple set-theoretic arguments shows that:

$$X \cap Y = \cup \mathcal{X} \cap \cup \mathcal{Y} = \bigcup_{Z \in \mathcal{X}, V \in \mathcal{Y}} Z \cap V.$$

Let $Z \in \mathcal{X}$ and $V \in \mathcal{Y}$. By the definition of a base, since $Z, V \in \mathcal{B}$, for each $x \in Z \cap V$, there is a set $U_x \in \mathcal{B}$ such that $x \in U_x \subseteq Z \cap V$. This means that $Z \cap V = \bigcup_{x \in Z \cap V} W_x$. This is true for each pair of sets Z, V with $Z \in \mathcal{X}$ and $V \in \mathcal{Y}$. Hence, since $X \cap Y$ is the union of sets of the form $Z \cap V$ where Z and V are both in $\mathcal{B}$ is the union of elements from $\mathcal{B}$, each of which is the union of elements from $\mathcal{B}$. Thus, $X \cap Y \in \mathcal{B}$. QED

Proposition 2.29 *For each* **S4** *neighborhood model $\mathfrak{M}$, there is topological model that is modally equivalent to $\mathfrak{M}$.*

[4] Recall that **S4** is the modal logic is sound and complete for topological models. We will see that the conditions definition and **S4** neighborhood frame characterize the axioms of **S4**.

Proof. Let $\mathfrak{M} = \langle W, N, V\rangle$ be an **S4** neighborhood model. The first step is to construct a topological model $\mathcal{M}^{T^N} = \langle W^N, \mathcal{T}^N, V^N\rangle$. Let $W^N = W$ and $V^N = V$. We first define a base $\mathcal{B}^N$ on W and let the topology $\mathcal{T}^N$ be generated by $\mathcal{B}^N$ using the construction from Lemma 2.28. Let $\mathcal{B}^N = \{m_N(X) \mid X \subseteq W\}$ (recall that $m_N(X) = \{v \mid X \in N(v)\}$). We will show that $\mathcal{B}^N$ is a base. That is, we must show that 1. $\cup\mathcal{B}^N = W$ and 2. for each $X, Y \in \mathcal{B}^N$ and each $x \in X \cap Y$ there is a $Z \in \mathcal{B}^N$ such that $x \in Z \subseteq X \cap Y$. The proof of (1.) follows directly from the assumption that each $N(w)$ is a consistent filter. To show (2.), Suppose that $X, Y \in \mathcal{B}^N$ and $x \in X \cap Y$. Then, $X = m_N(X_1)$ and $Y = m_N(X_2)$. Since $N(w)$ is a filter, $m_N(X_1) \cap m_N(X_2) = m_N(X_1 \cap X_2)$. Thus, $x \in m_N(X_1 \cap X_2) \subseteq m_N(X_1) \cap m_N(X_2)$. To complete the proof of the claim, we must show that $m_N(X_1 \cap X_2) \in \mathcal{B}^N$. But this follows from the third property in the definition of an **S4** neighborhood frame since $X_1 \cap X_2 \in N(w)$.

Let $\mathcal{T}^N$ be the topology generated from $\mathcal{B}^N$ using the construction from Lemma 2.28. We must show that for all formula $\varphi \in \mathcal{L}(\mathsf{At})$, $[\![\varphi]\!]_{\mathfrak{M}} = [\![\varphi]\!]_{\mathcal{M}^{T^N}}$. The proof is by induction on the structure of φ. The base case and Boolean connectives are as usual. We only show the case of the modal operators.

Suppose that $w \in [\![\Box\varphi]\!]_{\mathfrak{M}}$. Then, $[\![\varphi]\!]_{\mathfrak{M}} \in N(w)$, i.e., $w \in m_N([\![\varphi]\!]_{\mathfrak{M}})$. By condition 2 of the definition of an **S4** neighborhood frame, $m_N([\![\varphi]\!]_{\mathfrak{M}}) \subseteq [\![\varphi]\!]_{\mathfrak{M}}$ (why?). Hence, since $O = m_N([\![\varphi]\!]_{\mathfrak{M}}) \in \mathcal{T}^N$, and $w \in m_N([\![\varphi]\!]_{\mathfrak{M}}) \subseteq [\![\varphi]\!]_{\mathfrak{M}} = [\![\varphi]\!]_{\mathcal{M}^{T^N}}$ (the latter equality follows form the induction hypothesis), we have $w \in [\![\Box\varphi]\!]_{\mathcal{M}^{T^N}}$.

Suppose that $w \in\in [\![\Box\varphi]\!]_{\mathcal{M}^{T^N}}$. Then, there exists $O \in \mathcal{T}^N$ such that $w \in O \subseteq [\![\varphi]\!]_{\mathcal{M}^{T^N}}$. Since $O = \bigcup\mathcal{X}$ for some $\mathcal{X} \subseteq \mathcal{B}^N$, there is some $X \subseteq W$ such that $w \in m_N(X) \subseteq [\![\varphi]\!]_{\mathcal{M}^{T^N}} = [\![\varphi]\!]_{\mathfrak{M}}$ (the latter equality follows from the induction hypothesis). Then, $X \in N(w)$ and so, by condition 3 of the definition of an **S4** neighborhood frame, $m_N(X) \in N(w)$. Since $N(w)$ is closed under supersets and $m_N(X) \subseteq [\![\varphi]\!]_{\mathfrak{M}}$, we have $[\![\varphi]\!]_{\mathfrak{M}} \in N(w)$. Hence, $w \in [\![\Box\varphi]\!]_{\mathfrak{M}}$. QED

2.1.4 Plausibility Models

Originally used as a semantics for conditionals (cf. Lewis, 1973), *plausibility models* are widely used as a semantics for modal logics of belief (van Benthem, 2004; Baltag and Smets, 2006b,a; Girard, 2008). The main idea is to endow a set of states with an ordering $w \preceq v$ of relative *plausibility* on worlds: "(according to the agent) world v is at least as plausible as w". [5]

Definition 2.30 (Plausibility model) A **plausibility model** is a tuple $\mathcal{M} = \langle W, \preceq, V\rangle$ where W is a finite nonempty set, $\preceq \subseteq W \times W$ is a reflexive and transitive ordering on W, and $V : \mathsf{At} \to \wp(W)$ is a valuation function. If $\preceq$ is also *connected* (for each $w, v \in W$, either $w \preceq v$ or $v \preceq w$) then we say $\mathcal{M}$ is a **connected plausibility model**. A pair $\mathcal{M}, w$ where w is a state is called a **pointed (connected) plausibility model**. ◁

[5] In conditional semantics, plausibility or 'similarity' orders are typically world-dependent.

When two worlds w and v cannot be compared by a plausibility ordering, the interpretation is that the agent has either accepted contradictory evidence or lacks enough evidence to compare the two states.[6]

There are a number of different modal languages that have been used to reason about plausibility structures. For example, let $\mathcal{L}(\preceq, B, A)$ be the smallest set of formulas generated by the following grammar:

$$p \mid \neg\varphi \mid \varphi \wedge \psi \mid [B]\psi \mid [\preceq]\varphi \mid [A]\varphi$$

where $p \in \mathsf{At}$. For each $\bigcirc \in \{B, \preceq, A\}$, let $\langle\bigcirc\rangle\varphi$ be defined as $\neg[\bigcirc]\neg\varphi$. Before defining truth for this language, I need some notation. For $X \subseteq W$, let

$$Max_{\preceq}(X) = \{v \in X \mid \text{ there is no } v \in X \text{ such that } w \prec v\}$$

For each set X, $Max_{\preceq}(X)$ is the set of most plausible worlds in X (i.e., the maximal elements of X according to the plausibility order $\preceq$)[7]. Suppose that $\mathcal{M} = \langle W, \preceq, V\rangle$ is a plausibility model with $w \in W$. Truth of the boolean connectives and atomic propositions is defined as usual. I only give the clauses for the modal operators:

- $\mathcal{M}, w \models [B]\varphi$ iff $Max_{\preceq}(W) \subseteq [\![\varphi]\!]_{\mathcal{M}}$
- $\mathcal{M}, w \models [\preceq]\varphi$ iff for all $v \in W$, if $w \preceq v$ then $\mathcal{M}, v \models \varphi$
- $\mathcal{M}, w \models [A]\varphi$ iff for all $v \in W$, $\mathcal{M}, v \models \varphi$.

So, φ is believed provided φ is true throughout all of the most plausible states. There is much more to say about plausibility structures, their relationship with theories of belief revision, and the modal logic of beliefs (see van Benthem (2011) and Pacuit (2013) for discussions). In this Section, I focus on the relationship between plausibility models and neighborhood models.

From Plausibility Structures to Neighborhood Structures There is a natural subset space associated with every plausibility model.

Definition 2.31 (Upwards Closed Sets) Suppose that $\preceq$ is a plausibility ordering on a set of states W (i.e., a reflexive and transitive relation on W). The upwards closure of a set X, denoted $\uparrow X$, is the set

$$\uparrow X = \{v \in W \mid \text{there is a } w \in X \text{ such that } w \preceq v\}$$

A set X is **$\preceq$-closed** provided $\uparrow X \subseteq X$. Let $\mathcal{F}_{\preceq} = \{\uparrow X \mid X \subseteq W\}$ be the set of $\preceq$-closed sets. ◁

[6]Swanson (2011) has an extensive discussion of incomparability when modeling conditionals.

[7]To keep things simple, I assume that the set of worlds is finite, so this maximal set always exist. One needs a (converse) well-foundedness condition to guarantee this when there are infinitely many states.

Exercise 2.32 *If $\preceq$ is a plausibility ordering on W, then $\mathcal{F}_{\preceq}$ is closed under non-empty intersections. (Why do I need to specify non-empty intersections?) Is $\mathcal{F}_{\preceq}$ closed under supersets?*

Using the above notation, there is a straightforward way to turn any plausibility model into a neighborhood structure. Let $\mathcal{M} = \langle W, \preceq, V\rangle$ be a plausibility model. The associated neighborhood model in the model $\mathcal{M}^{\preceq} = \langle W^{\preceq}, N^{\preceq}, V^{\preceq}\rangle$ where $W^{\preceq} = W$; for each $w \in W$, $N^{\preceq}(w) = \mathcal{F}_{\preceq}$; and $V^{\preceq} = V$. Thus, the associated neighborhood model $\mathcal{M}^{\preceq}$ has a *uniform* neighborhood function (each state is associated with the same collection of sets).

Remark 2.33 *A more general definition of plausibility models is possible in which each state is associated with a different plausibility ordering. That is, for each $w \in W$, $\preceq_w$ is a plausibility ordering on W. In this case, the neighborhoods may vary at each state: $N^{\preceq}(w) = \mathcal{F}_{\preceq_w}$. I focus on a single, global plausibility ordering to simplify the discussion.*

The next step is to show that every plausibility model $\mathcal{M} = \langle W, \preceq, V\rangle$ is *equivalent* to the corresponding neighborhood model $\mathfrak{M}^{\preceq}$. Here we face a problem that did not arise in the previous sections. Note that the notion of equivalence between classes of models assumes that there is a single underlying language that can be interpreted on *both* classes of models. However, the problem is that the language $\mathcal{L}^{\preceq}$ has not been interpreted over neighborhood models.

One solution is to bite the bullet and give a definition of truth for a single modal language on both classes of models. In many cases, such a definition may not be available. Even in this case, it is still possible to relate two classes of models. I start by briefly discussing this approach. Let $\mathcal{L}(\langle\ ], A)$ be the propositional modal language generated by adding the neighborhood modality $\langle\ ]$ and the universal modality $[A]$ to a standard propositional language. [8]

The first observation is that, on finite models, we can restrict attention to the language containing only the modalities $[A]$ and $[\preceq]$. [9]

Fact 2.34 *On finite plausibility models, the belief modality $[B]$ is definable in terms of the $[A]$ and $[\preceq]$ modalities:*

- $B\varphi \;:=\; A\langle\preceq\rangle[\preceq]\varphi$

The proof is straightforward given the following observation. Note that the set of maximal elements in a plausibility model can be partitioned into *final clusters*:

Definition 2.35 (Final Cluster) Let $\mathcal{M} = \langle W, \preceq, V\rangle$ be a plausibility model. A **final cluster** in $\mathcal{M}$ is a set $X \subseteq Max_{\preceq}(W)$ that is completely connected: i.e., for any $x, y \in X$, $x \preceq y$ and $y \preceq x$, while no proper successors exist for worlds in X. $\triangleleft$

[8] Truth for formulas of the form $\langle\ ]\varphi$ is given in Section 1.3.1.
[9] This was first discussed by Boutilier (1992).

In a connected plausibility model, there is only one largest final cluster: the set $Max_{\preceq}(W)$. However, when the order is not connected, there may be more be disjoint final clusters. Using this terminology, (on finite models) $B\varphi$ is true provided φ is true throughout all final clusters. The key observation is that the combination $\langle\preceq\rangle[\preceq]$ refers to final clusters. Let $\mathcal{L}(\preceq, A)$ be the sublanguage of $\mathcal{L}(\preceq, A, B)$ without the belief modality $[B]$. I can now define the translation between languages and state the correspondence between models.

Definition 2.36 ($\preceq$-translation) The translation $tr_{\preceq} : \mathcal{L}(\langle\ ], A) \to \mathcal{L}(\preceq, A)$ is defined as follows:

- for each $p \in \mathsf{At}$, $tr_{\preceq}(p) = p$;
- $tr_{\preceq}(\neg\varphi) = \neg tr_{\preceq}(\varphi)$ and $tr_{\preceq}(\varphi \wedge \psi) = tr_{\preceq}(\varphi) \wedge tr_{\preceq}(\psi)$;
- $tr_{\preceq}(A\varphi) = A(tr_{\preceq}(\varphi))$; and
- $tr_{\preceq}(\langle\ ]\varphi) = \langle E\rangle[\preceq](tr_{\preceq}(\varphi))$. ◁

Proposition 2.37 *Let $\mathcal{M} = \langle W, \preceq, V\rangle$ be a plausibility model. For any $\varphi \in \mathcal{L}^{\langle\ ]}$ and state $w \in W$,*

$$\mathcal{M}, w \models tr_{\preceq}(\varphi) \text{ iff } \mathcal{M}^{\preceq}, w \models \varphi.$$

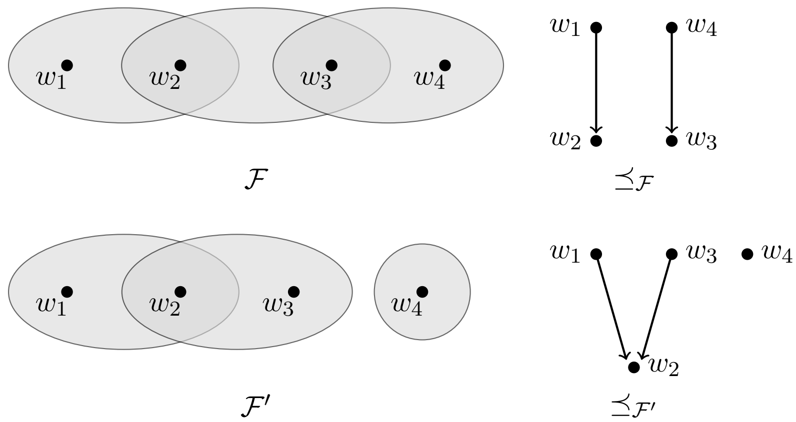
The proof is straightforward and left to the reader. However, this is a weak result. The conclusion is simply that every plausibility model "contains" an evidence model. Furthermore, the simple modal language $\mathcal{L}(\langle\ ], A)$ for reasoning about evidence models can be translated into the standard modal language $\mathcal{L}(\preceq, A)$ for reasoning about plausibility models. So, in short, *every plausibility model contains an evidence model.* It is more interesting to ask whether a plausibility ordering can be *derived* from an evidence function.

From Neighborhood Models to Plausibility Models There is also a natural way to define a plausibility ordering given any subset space. The approach is to use the so-called *specialization order*, a notion which occurs in point-set topology and in recent theories of relation merge (cf. Andreka et al., 2002; Liu, 2008).

Definition 2.38 (Specialization Order) Suppose that $\langle W, \mathcal{F}\rangle$ is a subset space. Define $\preceq_{\mathcal{F}} \subseteq W \times W$ as follows:

$$w \preceq_{\mathcal{F}} v \text{ iff for all } X \in \mathcal{F}, \text{ if } w \in X, \text{ then } v \in X \qquad ◁$$

The intuition is that y is "better" or "more special" than x provided every set in $\mathcal{F}$ that contains x also contains y. To make this definition a bit more concrete, here is a simple illustration.



Of course, the relational properties of $\preceq_{\mathcal{F}}$ depends on the algebraic properties of $\mathcal{F}$. However, all specialization orders are reflexive and transitive:

Observation 2.39 *Suppose that $\langle W, \mathcal{F}\rangle$ is a subset space. Then, $\preceq_{\mathcal{F}}$ is a transitive and reflexive relation on W.*

Proof. $w \preceq_F \; v$ and $v \preceq_{\mathcal{F}} y$. Let $X \in \mathcal{F}$. Suppose that $w \in X$. Then, $v \in X$ and so $y \in X$. Thus, $w \preceq_{\mathcal{F}} y$. QED

The examples given above show that, in general, the specialization order $\preceq_{\mathcal{F}}$ is not connected.

There are (at least) two different ways to associate a plausibility model with a neighborhood structure $\mathfrak{M} = \langle W, N, V\rangle$. The first is to assign to each state $w \in W$, a plausibility ordering $\preceq_{N(w)}$. Strictly speaking, unless N is a constant function, this is not a plausibility model according to Definition 2.30 since each state is assigned a different plausibility ordering (cf. Remark 2.33). Rather than pursuing this line of thought, I focus on the relationship between plausibility models and the evidence models from Section 1.5.1. It turns out that the precise relationship is subtle (see **?** and (van Benthem and Pacuit, 2011, Section 5) for discussions).

Definition 2.40 (Extended Evidence Model) An **extended evidence model** is a tuple $\mathcal{M} = \langle W, E, \preceq, B, V\rangle$, where W is a non-empty set of worlds; $E : W \to \wp(\wp(W))$ is an **evidence function**; $\preceq \subseteq W \times W$ is a plausibility order on W (i.e., a reflexive and transitive relation on W); $B \subseteq W \times W$ is an (arbitrary) binary relation on W; and $V : \mathsf{At} \to \wp(W)$ is a valuation function. The following constraints are satisfied:

1. for each $w \in W$, $\emptyset \notin E(w)$ and $W \in E(w)$;

2. for all $w, v, u \in W$, if $w \preceq v$ and $w \in X \in E(u)$, then $v \in X$; and

3. if $w \preceq v$ and $u \mathrel{B} w$, then $u \mathrel{B} v$. ◁

The earlier notions of *flat* and *uniform* evidence models can be adapted to extended evidence models:

Definition 2.41 (Flat, Uniform Extended Evidence Models) Suppose that $\mathcal{M} = \langle W, E, \preceq, B, V\rangle$ is an extended evidence model. The extended evidence model $\mathcal{M}$ is **flat** whenever $X \in E(w)$ there is $v \in X$ such that $w\ B\ v$.

An extended evidence model $\mathcal{M}$ is said to be **uniform** if E and B are constant (i.e., if there is some w, v such that wBv then for all states u, uBv and similarly for E) and whenever $w\ B\ v$ it follows that v is $\preceq$-maximal. In this case, I will treat E as a set $\mathcal{E}$ (of neighborhoods) rather than a function and B as a set of points.

Finally, $\mathcal{M}$ is **concise** if it is flat, uniform and if w is $\preceq$-maximal, then $w \in B$. ◁

Given an evidence model $\mathfrak{M} = \langle W, E, V\rangle$, define the extended model

$$\mathfrak{M}^{\triangle} = \langle W, E, B_E, \preceq_E, V\rangle.$$

where $w\ B_E\ v$ iff $v \in \bigcap \mathcal{X}$ for some w-scenario $\mathcal{X}$, and $\preceq_E = \preceq_{\bigcup_{w\in W} E(w)}$ (i.e., $w \preceq_E v$ iff for any u, X, if $w \in X \in E(u)$, then $v \in X$). The truth of formulas in $\mathcal{L}^{ev}$ is defined is the obvious way. I only give the clauses for the modal operators. Say that $\mathcal{M}$ is an **intended model** provided $\mathcal{M} = \langle W, E, V\rangle^{\triangle}$

I focus on the precise relationship between intended models $\mathcal{M}^{\triangle}$ and extended evidence models $\mathcal{M} = \langle W, E, B, \preceq, V\rangle$. First of all, notice that, in general, $\mathcal{M} = \langle W, E, B, \preceq_E, V\rangle$ is not necessarily a model according to Definition 2.40. The problem is that the constraint stating that if $w\ B\ v \preceq_E u$ then $w\ B\ u$ is not necessarily satisfied. A particularly simple example is a uniform model where $W = \{w, v\}$ and $E(w) = E(v) = \{W\}$. If, for example, $B = \{(w, w)\}$, then it should be clear that $w\ B\ w \preceq_E v$, yet $w \not B\ v$.

However, this can never happen on an intended model:

Lemma 2.42 *Suppose that $\mathfrak{M} = \langle W, E, V\rangle$ is an evidence model, then $\mathfrak{M}^{\triangle}$ is a model according to Definition 2.40.*

Proof. Most conditions are obvious; the only one that needs to be checked is that if $u\ B_E\ w$ and $w \preceq_E v$ then $u\ B_E\ v$. Since $u\ B_E\ w$, there is a scenario $\mathcal{X}$ such that $w \in \bigcap \mathcal{X}$. But since $w \preceq_E v$, it follows that also $v \in \bigcap \mathcal{X}$, and since $\mathcal{X}$ was a u-scenario, we have that $u\ B_E\ v$. QED

The following series of results characterizes the intended extended evidence models.

Lemma 2.43 *If $\mathcal{M} = \langle W, \mathcal{E}, B_{\mathcal{E}}, \preceq_{\mathcal{E}}, V\rangle$ is uniform and intended, then for every scenario $\mathcal{X}$ and every $w \in \bigcap \mathcal{X}$, w is $\preceq$-maximal if and only if w lies in $\bigcap \mathcal{X}'$ for some scenario $\mathcal{X}'$.*

Moreover, if $\mathcal{M}$ is flat then the sets of the form $\bigcap \mathcal{X}$ with $\mathcal{X}$ a scenario are precisely the $\preceq_{\mathcal{E}}$-equivalence classes of maximal worlds.

Proof. Let $\mathcal{X}$ be a scenario and $w \in \bigcap \mathcal{X}$. Suppose that w is $\preceq_{\mathcal{E}}$-maximal and let $\mathcal{X}'$ be a scenario extending $\mathcal{E}[w]$. We claim that $w \in \bigcap \mathcal{X}'$.

If this is not the case, then since $\mathcal{M}$ is flat, there is some $v \in \bigcap \mathcal{X}$. By definition $w \preceq v$, but since $w \notin \bigcap \mathcal{X}'$ there is some $Y \in \mathcal{E}[v] \setminus \mathcal{E}[w]$, so that $w \prec v$. This contradicts the maximality of w.

Conversely, if w lies in $\bigcap \mathcal{X}'$ for some scenario $\mathcal{X}'$ and $w \preceq v$, then we also have $v \in \bigcap \mathcal{X}'$. It follows that $\mathcal{X}' \cup \mathcal{E}[v]$ has the finite intersection property so $\mathcal{X}' = \mathcal{E}[v]$ by maximality and it follows that $v \preceq_{\mathcal{E}} w$ as well. QED

The plausibility orders in extended evidence models satisfy an additional property:

Definition 2.44 (Directed Plausibility Ordering) Let $\preceq$ be a plausibility order over W. Say $D \subseteq W$ is **directed** if any two elements of D have an upper bound in D.

A plausibility order $\preceq$ satisfies the **boundendess condition** if every directed set D has an upper bound (not necessarily in D). $\lhd$

Lemma 2.45 *If an evidence model is flat, then its derived plausibility relation satisfies the boundedness condition.*

Proof. Assume that $\mathfrak{M}$ is a flat evidence model and let D be any directed set. Consider the family $\mathcal{Y} = \{Y : Y \cap D \neq \varnothing\}$. We claim that $\mathcal{Y}$ has the f.i.p.. Indeed, let $Y_1, \ldots, Y_n \in \mathcal{Y}$ and $y_i \in Y_i$. Let y be an upper bound for all y_i (it is an easy exercise to show that directed sets have upper bounds for finite subsets). Then, by definition, $y \in \bigcap_{i \leq n} Y_i$.

Thus $\mathcal{Y}$ can be extended to a scenario $\mathcal{X}$. Since $\mathcal{W}$ is flat, then $\bigcap \mathcal{X}$ is non-empty. But every $w \in \bigcap \mathcal{X}$ is an upper bound for D. QED

Corollary 2.46 *If $\mathcal{M}$ is flat and $\preceq_E$ is its derived plausibility relation, then for every w there is v such that $w \preceq_E v$ and v is maximal.*

Proof. If $\preceq$ is directed it satisfies the conditions of Zorn's lemma, and we may use it to find maximal elements above any given w. QED

Finally, there is an analogue of Fact 2.34 showing that the belief modality can be defined in terms of the universal and plausibility modalities.

Theorem 2.47 *Over the class of uniform evidence models with derived plausibility relation, $[A]\langle\preceq\rangle[\preceq]\varphi$ implies $[B]\varphi$.*

Over the class of models that are moreover flat, the two formulas are equivalent.

Proof. First assume that $[A]\langle\preceq\rangle[\preceq]\varphi$ is true at w and let $\mathcal{X}$ be any w-scenario. Let $v \in \bigcap \mathcal{X}$. By Lemma 2.43, v is maximal; but since $\langle\preceq\rangle[\preceq]\varphi$ is true at v, it follows that v satisfies φ.

Now assume that the model is flat and $[B]\varphi$ holds at w and let $v \in W$. We have to show that there is v' such that $v \preceq v'$ and $[\preceq]\varphi$ is true at v'.

Use Lemma 2.43 to find a v'' such that $w \preceq v''$ and v'' is maximal. Then, also by Lemma 2.43 we have that v'' lies in $\bigcap \mathcal{X}$ for some w-scenario $\mathcal{X}$. Meanwhile, if $v'' \preceq u$ then we also have $u \in \bigcap \mathcal{X}$, and by the assumption that $[B]\varphi$ holds at w, u also satisfies φ, i.e. v satisfies $[\preceq]\varphi$, as desired. QED

2.2 Completeness

In this section, I show how to adapt the standard approach for proving completeness of modal logics to prove completeness of non-normal modal logics with respect to neighborhood semantics. I assume that the reader is familiar with basic soundness and completeness results in modal logic (with respect to relational frames), see Blackburn et al. (2001) for more information. In Section 2.2.1, I review some basic terminology. The proofs of completeness for the key systems of non-normal modal logic are found in Section **??**. Section **??** discusses completeness for neighborhood models with a universal modality. Finally, Section **??** discuss an example of a consistent modal logic that is not complete with respect to any class of neighborhood frames.

2.2.1 Preliminaries

Let F be a collection of neighborhood frames. A formula $\varphi \in \mathcal{L}$ is **valid in** F, or F**-valid**, if for each $\mathfrak{F} \in \mathsf{F}$, $\mathfrak{F} \models \varphi$. Given a class of frame F, let $\mathbf{L}(\mathsf{F}) = \{\varphi \mid \text{for all } \mathfrak{F} \in \mathsf{F}, \mathfrak{F} \models \varphi\}$ denote the set of formulas that are F-valid.

Definition 2.48 (Soundness) A logic $\mathbf{L}$ is **sound** with respect to F, provided $\mathbf{L} \subseteq \mathbf{L}(\mathsf{F})$. That is, for each $\varphi \in \mathcal{L}$, if $\vdash_{\mathbf{L}} \varphi$, then φ is valid in F. ◁

The key concept that we will study in this section is semantic consequence:

Definition 2.49 (Semantic Consequence) Suppose that Γ is a set of formulas, $\varphi \in \mathcal{L}$ is a formula, and F is a class of neighborhood frames. The formula φ is a **semantic consequence** with respect to F of Γ, denoted $\Gamma \models_{\mathsf{F}} \varphi$, provided for each $\mathfrak{F} \in \mathsf{F}$, if $\mathfrak{F} \models \Gamma$, then $\mathfrak{F} \models \varphi$. Here $\mathfrak{F} \models \Gamma$ means that for each $\varphi \in \Gamma$, $\mathfrak{F} \models \varphi$. ◁

One final bit notion: write $\models_{\mathsf{F}} \varphi$ if for each $\mathfrak{F} \in \mathsf{F}$, $\mathfrak{F} \models \varphi$.

Definition 2.50 (Weak, Strong Completeness) A logic $\mathbf{L}$ is **weakly complete** with respect to a class of frames F, if $\models_{\mathsf{F}} \varphi$ implies $\vdash_{\mathbf{L}} \varphi$. A logic $\mathbf{L}$ is **strongly complete** with respect to a class of frames F, if for each set of formulas Γ, $\Gamma \models_{\mathsf{F}} \varphi$ implies $\Gamma \vdash_{\mathbf{L}} \varphi$. ◁

Let $\mathbf{L}$ be any modal logic. A set of formulas Γ is said to be **L-inconsistent** if $\Gamma \vdash_{\mathbf{L}} \bot$. The set Γ is **L-consistent** if it is not inconsistent.

Definition 2.51 (Maximally Consistent Set) A set of formulas Γ is called a **maximally consistent set** provided Γ is a consistent set of formulas and for all formulas $\varphi \in \mathcal{L}$, either $\varphi \in \Gamma$ or $\neg\varphi \in \Gamma$. ◁

Let $M_{\mathbf{L}}$ be the set of **L**-maximally consistent sets of formulas. Given a formula $\varphi \in \mathcal{L}$, let $|\varphi|_{\mathbf{L}}$ be the **proof set** of φ in **L**. Formally, $|\varphi|_{\mathbf{L}} = \{\Delta \mid \Delta \in M_{\mathbf{L}} \text{ and } \varphi \in \Delta\}$. We first note that proof sets share a number of properties in common with truth sets.

Lemma 2.52 *Let* **L** *be a logic and* $\varphi, \psi \in \mathcal{L}$. *Then*

1. $|\varphi \wedge \psi|_{\mathbf{L}} = |\varphi|_{\mathbf{L}} \cap |\psi|_{\mathbf{L}}$
2. $|\neg\varphi|_{\mathbf{L}} = M_{\mathbf{L}} - |\varphi|_{\mathbf{L}}$
3. $|\varphi \vee \psi|_{\mathbf{L}} = |\varphi|_{\mathbf{L}} \cup |\psi|_{\mathbf{L}}$
4. $|\varphi|_{\mathbf{L}} \subseteq |\psi|_{\mathbf{L}}$ *iff* $\vdash_{\mathbf{L}} \varphi \rightarrow \psi$
5. $|\varphi|_{\mathbf{L}} = |\psi|_{\mathbf{L}}$ *iff* $\vdash_{\mathbf{L}} \varphi \leftrightarrow \psi$
6. *For any maximally* **L***-consistent set* Γ, *if* $\varphi \in \Gamma$ *and* $\varphi \rightarrow \psi \in \Gamma$, *then* $\psi \in \Gamma$
7. *For any maximally* **L***-consistent set* Γ, *If* $\vdash_{\mathbf{L}} \varphi$, *then* $\varphi \in \Gamma$

Proof. The proofs are standard facts about maximally consistent sets and left for the reader. QED

Another standard result is *Lindenbaum's Lemma* (see Chellas (1980) and Blackburn et al. (2001) for a proof and extended discussion).

Lemma 2.53 (Lindenbaum's Lemma) *For any consistent set of formulas* Γ, *there exists a maximally consistent set* Γ' *such that* $\Gamma \subseteq \Gamma'$.

The following useful fact about proof sets demonstrates how Lindenbaum's Lemma can be used.

Lemma 2.54 *For each* $\varphi \in \mathcal{L}$, $\psi \in \bigcap |\varphi|_{\mathbf{L}}$ *iff* $\vdash_{\mathbf{L}} \varphi \rightarrow \psi$.

Proof. Suppose that $\vdash_{\mathbf{L}} \varphi \rightarrow \psi$. Then, for each maximally consistent set Γ, $\varphi \rightarrow \psi \in \Gamma$. Hence, since for each $\Gamma \in |\varphi|_{\mathbf{L}}$, $\varphi \in \Gamma$, we have $\psi \in \Gamma$. Thus $\varphi \in \cap|\varphi|_{\mathbf{L}}$.

Suppose that $\psi \in \cap|\varphi|_{\mathbf{L}}$ but it is not the case that $\vdash_{\mathbf{L}} \varphi \rightarrow \psi$. Then $\neg(\varphi \rightarrow \psi)$ is **L**-consistent. Using Lindenbaum's Lemma, there is a maximally consistent set Γ such that $\neg(\varphi \rightarrow \psi) \in \Gamma$. Thus, $\varphi, \neg\psi \in \Gamma$. Since $\varphi \in \Gamma$, $\Gamma \in |\varphi|_{\mathbf{L}}$. But then $\neg\psi \in \Gamma$ contradicts the fact that $\psi \in \cap|\varphi|_{\mathbf{L}}$. QED

2.2.2 The Proofs

Suppose that $\mathfrak{M} = \langle W, N, V\rangle$ is a neighborhood model and $X \subseteq W$ any subset. A set $X \subseteq$ is **definable** (with respect to a modal language $\mathcal{L}$) provide there is a formula $\varphi \in \mathcal{L}$ such that $[\![\varphi]\!]_{\mathfrak{M}} = X$. Let $\mathcal{D}_{\mathfrak{M}}$ be the set of all sets that are definable in $\mathfrak{M}$. Note that since there are only countably many formulas in $\mathcal{L}$, the set $\mathcal{D}_{\mathfrak{M}}$ is countable (or finite if W is finite). Thus, if W is countable, since $\wp(W)$ is uncountable, it may be the case that $\mathcal{D}_{\mathfrak{M}} \neq \wp(W)$. A subset $X \subseteq M_{\mathbf{L}}$ is called a **proof set** provided there is some formula $\varphi \in \mathcal{L}$ such that $X = |\varphi|_{\mathbf{L}}$. Again notice that there are only countably many proof sets; however, if At is countable, then $M_{\mathbf{L}}$ is uncountable, and hence there are uncountably many subsets of $M_{\mathbf{L}}$.

As usual, the states in a canonical model will be maximally consistent sets, i.e., elements of $M_{\mathbf{L}}$. Any function $N_{\mathbf{L}} : M_{\mathbf{L}} \to \wp(\wp(M_{\mathbf{L}}))$ is a canonical neighborhood function provided for all $\varphi \in \mathcal{L}$:

$$|\varphi|_{\mathbf{L}} \in N_{\mathbf{L}}(\Gamma) \text{ iff } \Box\varphi \in \Gamma.$$

So, for each maximally consistent set Γ, $N_{\mathbf{L}}(\Gamma)$ contains at least all the proof sets of the necessary formulas from Γ. The first question is *do any such functions actually exist?* That is, is it even possible to define a function satisfying the above condition? A problem would arise if there are proof sets $|\varphi|_{\mathbf{L}}$ and $|\psi|_{\mathbf{L}}$ such that $|\varphi|_{\mathbf{L}} = |\psi|_{\mathbf{L}}$ and a maximally consistent set with $\Box\varphi \in \Gamma$ but $\Box\psi \notin \Gamma$. If such a situation is possible, then it would be impossible to satisfy the above condition. Fortunately, this problematic situation is ruled out in any logic contain the rule *RE*.

Lemma 2.55 *Suppose that* $\mathbf{L}$ *is a logic in which the rule RE is admissible and that* $N_{\mathbf{L}} : M_{\mathbf{L}} \to \wp(\wp(M_{\mathbf{L}}))$ *is a function such that for each* $\Gamma \in M_{\mathbf{L}}$, $|\varphi|_{\mathbf{L}} \in N_{\mathbf{L}}(\Gamma)$ *iff* $\Box\varphi \in \Gamma$. *Then, if* $|\varphi|_{\mathbf{L}} \in N_{\mathbf{L}}(\Gamma)$ *and* $|\varphi|_{\mathbf{L}} = |\psi|_{\mathbf{L}}$, *then* $\Box\psi \in \Gamma$.

Proof. Let φ and ψ be two formulas such that $|\varphi|_{\mathbf{L}} = |\psi|_{\mathbf{L}}$, $\Box\varphi \in \Gamma$, and $|\varphi|_{\mathbf{L}} \in N_{\mathbf{L}}(\Gamma)$. Since $|\varphi|_{\mathbf{L}} \in N_{\mathbf{L}}(\Gamma)$, $\Box\varphi \in \Gamma$. Also, by Lemma 2.52, since $|\varphi|_{\mathbf{L}} = |\psi|_{\mathbf{L}}$, $\vdash_{\mathbf{L}} \varphi \leftrightarrow \psi$. Using the *RE* rule, $\vdash_{\mathbf{L}} \Box\varphi \leftrightarrow \Box\psi$. Hence, $\Box\varphi \leftrightarrow \Box\psi \in \Gamma$. Therefore, $\Box\psi \in \Gamma$. QED

The **canonical valuation**, $V_{\mathbf{L}} : \mathsf{At} \to \wp(M_{\mathbf{L}})$, is defined as follows. For each $p \in \mathsf{At}$, let $V_{\mathbf{L}}(p) = |p|_{\mathbf{L}} = \{\Gamma \mid \Gamma \in M_{\mathbf{L}} \text{ and } p \in \Gamma\}$. Putting everything together gives us the following definition:

Definition 2.56 (Canonical Neighborhood Model) A neighborhood model $\mathfrak{M} = \langle W, N, V\rangle$ is said to be **canonical for L** provided:

1. $W = M_{\mathbf{L}}$;
2. for each $\Gamma \in W$ and each formula $\varphi \in \mathcal{L}$, $|\varphi|_{\mathbf{L}} \in N(\Gamma)$ iff $\Box\varphi \in \Gamma$; and
3. $V = V_{\mathbf{L}}$. $\triangleleft$

For example, let $\mathfrak{M}_{\mathbf{L}}^{min} = \langle M_{\mathbf{L}}, N_{\mathbf{L}}^{min}, V_{\mathbf{L}}\rangle$ with for each $\Gamma \in M_{\mathbf{L}}$, $N_{\mathbf{L}}^{min}(\Gamma) = \{|\varphi|_{\mathbf{L}} \mid \Box\varphi \in \Gamma\}$. The model $\mathfrak{M}_{\mathbf{L}}^{min}$ is easily seen to be canonical for $\mathbf{L}$. Furthermore, it is the minimal canonical for $\mathbf{L}$ in the sense that for each Γ, $N_{\mathbf{L}}(\Gamma)$ is the smallest set satisfying the required property. Let $\mathcal{P}_{\mathbf{L}}$ be the set of all proof sets of $\mathbf{L}$ (i.e., $\mathcal{P}_{\mathbf{L}} = \{|\varphi|_{\mathbf{L}} \mid \varphi \in \mathcal{L}\}$). The largest canonical for $\mathbf{L}$ is the model $\mathfrak{M}_{\mathbf{L}}^{max} = \langle M_{\mathbf{L}}, N_{\mathbf{L}}^{max}, V_{\mathbf{L}}\rangle$ with for each $\Gamma \in M_{\mathbf{L}}$, $N_{\mathbf{L}}^{max}(\Gamma) = N_{\mathbf{L}}^{min}(\Gamma) \cup \{X \mid X \subseteq M_{\mathbf{L}}, X \not\in P_{\mathbf{L}}\}$.

Lemma 2.57 (Truth Lemma) *For any consistent logic* $\mathbf{L}$ *and any consistent formula* φ*, if* $\mathfrak{M}$ *is canonical for* $\mathbf{L}$*, then*

$$[\![\varphi]\!]_{\mathfrak{M}} = |\varphi|_{\mathbf{L}}$$

Proof. Suppose that $\mathfrak{M} = \langle W, N, V\rangle$ is a canonical model for $\mathbf{L}$. The proof is by induction on the structure of $\varphi \in \mathcal{L}$. The base case and the cases for the Boolean connectives are as usual and left for the reader. I only give the details for the modal case. Suppose that $\Gamma \in [\![\Box\varphi]\!]_{\mathfrak{M}}$. Then, by the definition of truth, $[\![\varphi]\!]_{\mathfrak{M}} \in N(\Gamma)$. By the induction hypothesis, $[\![\varphi]\!]_{\mathfrak{M}} = |\varphi|_{\mathbf{L}}$; hence, $|\varphi|_{\mathbf{L}} \in N(\Gamma)$. By part 2 of Definition 2.56, $\Box\varphi \in \Gamma$. Hence $\Gamma \in |\Box\varphi|_{\mathbf{L}}$. Conversely, suppose that $\Gamma \in |\Box\varphi|_{\mathbf{L}}$. Then by definition of a truth set, $\Box\varphi \in \Gamma$. Hence, by part 2 of Definition 2.56, $|\varphi|_{\mathbf{L}} \in N(\Gamma)$. By the induction hypothesis, $|\varphi|_{\mathbf{L}} = [\![\varphi]\!]_{\mathfrak{M}}$; hence, $[\![\varphi]\!]_{\mathfrak{M}} \in N(\Gamma)$. Therefore, $\Gamma \in [\![\Box\varphi]\!]_{\mathfrak{M}}$. QED

Theorem 2.58 *The logic* $\mathbf{E}$ *is sound and strongly complete with respect to the class of all neighborhood frames.*

Proof. The proof is standard and so will only be sketched (see Blackburn et al. (2001) for a discussion). Soundness is straightforward (and in fact already shown in earlier exercises). As for strong completeness, we will show that every consistent set of formulas can be satisfied in some model. Before proving this, we briefly explain why this implies strong completeness. The proof is by contraposition. Suppose that it is not the case that $\Gamma \vdash_{\mathbf{L}} \varphi$. Then $\Gamma \cup \{\neg\varphi\}$ is consistent. If this set is jointly true at some state in a model, then then Γ cannot semantically entail φ. Thus, if $\Gamma \not\vdash_{\mathbf{L}} \varphi$ then $\Gamma \not\models_{\mathsf{F}} \varphi$ (where F is the class of all neighborhood frames).

Let Γ be a consistent set of formulas. By Lindenbaum's Lemma, there is a maximally consistent set Γ' such that $\Gamma \subseteq \Gamma'$. Consider the minimal canonical model $\mathfrak{M}_{\mathbf{E}}^{min}$. By the Truth Lemma (Lemma 2.57), $\mathfrak{M}_{\mathbf{L}}^{min}, \Gamma' \models \Gamma'$. Thus Γ is true at a state in the minimal canonical model, as desired. QED

Notice that in the above proof, the choice to use the *minimal* canonical model for $\mathbf{E}$ was somewhat arbitrary. It is easy to see that the proof would go through if we had used $\mathfrak{M}_{\mathbf{E}}^{max}$ instead of $\mathfrak{M}_{\mathbf{E}}^{min}$. Indeed, *any* canonical model for $\mathbf{E}$ could have been used in the above proof. The fact that there is a choice of canonical models is very useful will be useful when proving completeness of systems logics above $\mathbf{E}$. , will be of great use when proving completeness of systems above $\mathbf{E}$. The strategy for proving strong completeness for the other systems of non-normal modal logics discussed in Section **??** is similar to the strategy for proving strong completeness of some well-known normal modal logics, such as **S4** or **S5**. Given the above definition of a canonical model and truth lemma, all that remains is to show that a the frame of a particular canonical model belongs to the class of frames under consideration. This argument is called *completeness-via-canonicity* in (Blackburn et al., 2001). For instance, consider the logic **EC**. We argued in the previous section that **C** corresponds to the neighborhoods being closed under (finite) intersections. We now show that **EC** is sound and strongly complete with respect to neighborhood frames that are closed under intersections. We first show that C is *canonical* for this property (see Blackburn et al. (2001) Chapter 4 for an extended discussion of this notion).

Lemma 2.59 *If* **L** *contains all instances of* C*, then* $N^{min}_{\mathbf{L}}$ *is closed under finite intersections.*

Proof. Suppose that **L** contains all instances of C. We must shows that for all $\Gamma \in M_{\mathbf{L}}$, $N^{min}_{\mathbf{L}}(\Gamma)$ is closed under intersections. Suppose that $X, Y \in N^{min}_{\mathbf{L}}(\Gamma)$. By the definition of $N^{min}_{\mathbf{L}}$, $X = |\varphi|_{\mathbf{L}}$ and $Y = |\psi|_{\mathbf{L}}$ with $\Box\varphi \in \Gamma$ and $\Box\psi \in \Gamma$. Hence $\Box\varphi \wedge \Box\psi \in \Gamma$; and so, using C, $\Box(\varphi \wedge \psi) \in \Gamma$. Thus, $|\varphi \wedge \psi|_{\mathbf{L}} \in N^{min}_{\mathbf{L}}(\Gamma)$. Therefore, $X \cap Y = |\varphi|_{\mathbf{L}} \cap |\psi|_{\mathbf{L}} = |\varphi \wedge \psi|_{\mathbf{L}} \in N^{min}_{\mathbf{L}}(\Gamma)$, as desired. QED

Given the above proof, strong completeness is straightforward.

Theorem 2.60 *The logic* **EC** *is sound and strongly complete with respect to the class of neighborhood frames that are closed under intersections.*

Proof. The proof is left as an exercise for the reader. QED

Exercise 2.61 *Prove that* **EN** *is sound and strongly complete with respect to neighborhood frames that contain the unit.*

The proof that **EM** is strongly complete with respect to neighborhood frames that are closed under supersets is not as straightforward. It is here where we need to make use of the fact that there are a number of different canonical models. The main difficulty is that $N^{min}_{\mathbf{EM}}$ is not closed under supersets.

Observation 2.62 *There is a maximally consistent set* Γ *such that* $N^{min}_{\mathbf{EM}}(\Gamma)$ *is not closed under supersets.*

Proof. Let p be a propositional variable and let Γ be a maximally consistent set such that $\Box p \in \Gamma$ (such a set exists by Lindenbaum's Lemma since $\Box p$ is consistent). Then $|p|_{\mathbf{EM}} \in N^{min}_{\mathbf{EM}}(\Gamma)$. Let Y be any non-proof set that extends $|p|_{\mathbf{EM}}$. To see that such a set exists, let Y' be any non-proof set (such a set exists since there are uncountably many subsets of $M_{\mathbf{EM}}$ but only countably many proof sets.) Then, $Y = Y' \cup |p|_{\mathbf{EM}}$ is not a proof set. For if $Y = |\psi|_{\mathbf{EM}}$ for some formula ψ, then $Y' = |\psi \wedge \neg p|_{\mathbf{EM}}$ (why?), which contradicts the fact that Y' is not a proof set. Clearly, $Y \notin N^{min}_{\mathbf{EM}}(\Gamma)$ (why?). Then, we have $X = |p|_{\mathbf{EC}} \in N^{min}_{\mathbf{EM}}(\Gamma)$, $X \subseteq Y$, but $Y \notin N^{min}_{\mathbf{EM}}(\Gamma)$. QED

However, this difficulty can be easily overcome by choosing a different, better behaved, canonical model. Recall from Section 2, that if $\mathcal{F}$ is any collection of subsets of W, then $sup(\mathcal{F}) = \{X \mid \exists Y \in \mathcal{F} \text{ where } Y \subseteq X\}$. Given any model $\mathfrak{M} = \langle W, N, V\rangle$, let the **supplementation** of $\mathfrak{M}$, denoted $sup(\mathfrak{M})$, be the model $\langle W, N^{sup}, V\rangle$, where for each $w \in W$, $N^{sup}(w) = sup(N(w))$. The key argument is that the supplementation of the minimal canonical model is canonical for **EM**.

Lemma 2.63 *Suppose that* $\mathfrak{M} = sup(\mathfrak{M}^{min}_{\mathbf{EM}})$*. Then* $\mathfrak{M}$ *is canonical for* **EM**.

Proof. Suppose that $\mathfrak{M} = \langle W, N, V\rangle$, where $W = M_{\mathbf{EM}}$ and for each $\Gamma \in W$, $N(\Gamma) = sup(N^{min}_{\mathbf{EM}}(\Gamma)$, and $V = V_{\mathbf{EM}}$. Let $\Gamma \in W$. We must show that for each formula $\varphi \in \mathcal{L}$,

$$|\varphi|_{\mathbf{EM}} \in N(\Gamma) \text{ iff } \Box\varphi \in \Gamma$$

The right to left direction is trivial since for all Γ, $N^{min}_{\mathbf{EM}}(\Gamma) \subseteq N(\Gamma)$. Suppose that $|\varphi|_{\mathbf{EM}} \in N(\Gamma) = sup(N^{min}_{\mathbf{EM}}(\Gamma))$. Then, there is some proof set $|\psi|_{\mathbf{EM}} \in N^{min}_{\mathbf{EM}}(\Gamma)$ such that $|\psi|_{\mathbf{EM}} \subseteq |\varphi|_{\mathbf{EM}}$. Since $|\psi|_{\mathbf{EM}} \in N^{min}_{\mathbf{EM}}(\Gamma)$, we have $\Box\psi \in \Gamma$. Furthermore, since $|\psi|_{\mathbf{EM}} \subseteq |\varphi|_{\mathbf{EM}}$, by Lemma **??**, $\vdash_{\mathbf{EM}} \varphi \rightarrow \psi$. Using the rule RM (which is admissible in **EM**), $\vdash_{\mathbf{EM}} \Box\psi \rightarrow \Box\varphi$. Thus, $\Box\psi \rightarrow \Box\varphi \in \Gamma$. Therefore, $\Box\varphi \in \Gamma$, as desired. QED

Theorem 2.64 *The logic* **EM** *is sound and strongly complete with respect to the class of supplemented frames.*

Proof. Left as an exercise for the reader. QED

Putting everything together gives a characterization of the smallest normal modal logic **K**.

Theorem 2.65 *The logic* **K** *is sound and strongly complete with respect to the class of filters.*

Exercise 2.66 *Prove that* **K** *is sound and strongly complete with respect to the class of augmented frames.*

Exercise 2.67 *Find an axiomatization and prove soundness and completeness for modal logics with the following modalities:* $[\ \rangle, \langle\ ], [\ ], \langle\ \rangle$

2.2.3 Non-Normal Modal Logic with the Universal Modality

The method for proving completeness discussed in the previous section can be adapted to deal with logics that include both normal and non-normal modalities. Of course, the situation is most interesting when there is non-trivial interaction between the different modalities. In this subsection, I discuss completeness for such a modal logic which includes both a non-normal modality and the *universal modality*[10] Let $\mathcal{L}^{MA}$ be the smallest set of formulas generated by the following grammar:

$$p \mid \neg\varphi \mid \varphi \wedge \psi \mid \langle\,]\varphi \mid A\varphi$$

where $p \in \mathsf{At}$. The additional Boolean connectives are defined as usual. Furthermore, let $[\,\rangle\varphi$ be defined as $\neg\langle\,]\neg\varphi)$ and $E\varphi$ defined as $\neg A\neg\varphi$. Formulas of $\mathcal{L}^{MA}$ are interpreted on neighborhood models $\mathfrak{M} = \langle W, N, V\rangle$. As a reminder, the truth clauses for the two modalities are:

- $\mathfrak{M}, w \models \langle\,]\varphi$ iff there exists $X \in N(w)$ such that $X \subseteq [\![\varphi]\!]_{\mathfrak{M}}$.
- $\mathfrak{M}, w \models A\varphi$ iff for all $v \in W$, $\mathcal{M}, v \models \varphi$.

Let **EMA** be the logic consisting of the following axiom schemes and rules:

$(A\text{-}K)$	$A(\varphi \to \psi) \to (A\varphi \to A\psi)$
$(A\text{-}T)$	$A\varphi \to \varphi$
$(A\text{-}4)$	$A\varphi \to AA\varphi$
$(A\text{-}B)$	$E\varphi \to AE\varphi$
$(A\text{-}Nec)$	From φ infer $A\varphi$
$(\langle\,]\text{-}RM)$	From $\varphi \to \psi$ infer $\langle\,]\varphi \to \langle\,]\psi$
$(\langle\,]\text{-}Cons)$	$\neg\langle\,]\bot$
$(A\text{-}N)$	$A\varphi \to \langle\,]\varphi$
$(Pullout)$	$\langle\,](\varphi \wedge A\psi) \leftrightarrow (\langle\,]\varphi \wedge A\psi)$

Lemma 2.68 *The axiom Pullout is valid on any neighborhood model $\mathfrak{M} = \langle W, N, V\rangle$ in which for all $w \in W$, $\emptyset \notin N(w)$.*

Proof. As the reader is invited to verify, for any formula α and any neighborhood model $\mathfrak{M} = \langle W, N, V\rangle$, we have

$$[\![A\alpha]\!]_{\mathfrak{M}} = \begin{cases} W & [\![\alpha]\!]_{\mathfrak{M}} = W \\ \emptyset & [\![\alpha]\!]_{\mathfrak{M}} \neq W \end{cases}$$

Let $\mathfrak{M} = \langle W, N, V\rangle$ be a neighborhood model such that for all $w \in W$, $\emptyset \notin N(w)$. We must show that $[\![\langle\,](\varphi \wedge A\psi)]\!]_{\mathfrak{M}} = [\![\langle\,]\varphi \wedge A\psi]\!]_{\mathfrak{M}}$. There are two cases:

1. $[\![\psi]\!]_{\mathfrak{M}} = W$. Then, $[\![A\psi]\!]_{\mathfrak{M}} = W$. Since $[\![\varphi]\!]_{\mathfrak{M}} \cap [\![A\psi]\!]_{\mathfrak{M}} = [\![\varphi]\!]_{\mathfrak{M}}$, we have $[\![\langle\,](\varphi \wedge A\psi)]\!]_{\mathfrak{M}} = [\![\langle\,]\varphi]\!]_{\mathfrak{M}} = [\![\langle\,]\varphi]\!]_{\mathfrak{M}} \cap [\![A\psi]\!]_{\mathfrak{M}} = [\![\langle\,]\varphi \wedge A\psi]\!]_{\mathfrak{M}}$.

[10] See (**?**) for an extension discussion of modal logics that include a universal modality.

2. $[\![\psi]\!]_{\mathfrak{M}} = \emptyset$. Then, $[\![A\psi]\!]_{\mathfrak{M}} = \emptyset$. Furthermore, since $\emptyset \not\in N(w)$ for any w, we have $[\![\langle\ ](\varphi \wedge A\psi)]\!]_{\mathfrak{M}} = \emptyset = [\![\langle\ ]\varphi]\!]_{\mathfrak{M}} \cap \emptyset = [\![\langle\ ]\varphi]\!]_{\mathfrak{M}} \cap [\![A\psi]\!]_{\mathfrak{M}} = [\![\langle\ ]\varphi \wedge A\psi]\!]_{\mathfrak{M}}$.

QED

Exercise 2.69 *Prove that A-N is valid on any neighborhood model* $\mathfrak{M} = \langle W, N, V\rangle$ *in which for all* $w \in W$, $W \in N(w)$.

The completeness proof combines ideas from the previous section and the standard method for proving completeness with respect to relational model. Let $\mathsf{F}_{EMA} = \{\mathfrak{F} \mid \mathfrak{F} = \langle W, N\rangle$ with for all $w \in W$, $\emptyset \not\in N(w)$ and $W \in N(w)\}$. Recall that $M_{\mathbf{EMA}}$ be the set of maximally **EMA**-consistent sets. Finally, suppose that R^A is a relation on $M_{\mathbf{EMA}}$ defined as follows:

$$\Gamma\ R^A\ \Delta \text{ iff } \Gamma^A = \{\varphi \mid A\varphi \in \Gamma\} \subseteq \Delta.$$

The following Lemma is a consequence of the fact that **EMA** contains the axiom schemes *A-T*, *A*-4 and *A-B* (see Blackburn et al., 2001, for details).

Lemma 2.70 *The relation* R^A *is an equivalence relation.*

The proof of this standard fact about modal logic is left to the reader. Let Γ be a maximally consistent set. For any maximally consistent set $\Gamma \in M_{\mathbf{EMA}}$, construct a canonical model $\mathfrak{M}^\Gamma = \langle W^\Gamma, N^\Gamma, V^\Gamma\rangle$ as follows:

- $W^\Gamma = R^A(\Gamma) = \{\Delta \mid \Gamma\ R^A\ \Delta\}$;
- For each $\Delta \in W^\Gamma$, $N^\Gamma(\Delta) = \{|\varphi|_{\mathbf{EMA}} \cap W^\Gamma \mid \Box\varphi \in \Delta\}$; and
- For all $p \in \mathsf{At}$, $V^\Gamma(p) = \{\Delta \in W^\Gamma \mid p \in \Delta\} = |p|_{\mathbf{EMA}} \cap W^\Gamma$.

Before proving a Truth Lemma for this canonical model, we need a preliminary result about the logic **EMA**.

Lemma 2.71 *Suppose that* Γ *is a set of formulas. If* $\Gamma, \varphi \vdash_{\mathbf{EMA}} \psi$, *then* $A\Gamma, \Box\varphi \vdash_{\mathbf{EMA}} \psi$, *where* $A\Gamma = \{A\varphi \mid \varphi \in \Gamma\}$.

Proof. Without loss of generality we can replace Γ with a single formula γ (why?). Suppose that $\gamma, \varphi \vdash_{\mathbf{EMA}} \psi$. Then, since $A\gamma \to \gamma$, we also have $A\gamma \wedge \varphi \vdash_{\mathbf{EMA}} \psi$. Thus, $A\gamma \wedge \varphi \vdash_{\mathbf{EMA}} \psi$ By *RM*, $\langle\ ](A\gamma \wedge \varphi) \vdash_{\mathbf{EMA}} \langle\ ]\psi$. Using the axiom (*Pullout*), we have $A\gamma \wedge \langle\ ]\varphi \vdash_{\mathbf{EMA}} \langle\ ]\psi$, as desired. QED

Lemma 2.72 (Truth Lemma) *Suppose that* Γ *is a maximally consistent set. For any formula* $\varphi \in \mathcal{L}^{MA}$,

$$[\![\varphi]\!]_{\mathfrak{M}^\Gamma} = |\varphi|_{\mathbf{EMA}} \cap R^A(\Gamma).$$

Proof. Suppose that Γ is a maximally consistent set and $\mathfrak{M}^\Gamma$ is the canonical model for Γ. The proof is by induction on the structure of $\varphi \in \mathcal{L}^{MA}$. The base case is a direct consequence of the definition of the canonical valuation:

$$[\![p]\!]_{\mathfrak{M}^\Gamma} = V^\Gamma(p) = |p|_{\mathbf{EMA}} \cap R^A(\Gamma).$$

The proof for the Boolean connectives are straightforward. I only give the details for the modal operators:

- ($[\![\langle\ ]\varphi]\!]_{\mathfrak{M}^\Gamma} = |\langle\ ]\varphi|_{\mathbf{EMA}} \cap R^A(\Gamma)$): Suppose that $\Delta \in |\langle\ ]\varphi|_{\mathbf{EMA}} \cap R^A(\Gamma)$. Then, $\langle\ ]\varphi \in \Delta$ and $\Delta \in R^A(\Gamma)$. By construction, $|\varphi| \cap R^A(\Gamma) \in N^\Gamma(\Delta)$. Hence, by the induction hypothesis, $[\![\varphi]\!]_{\mathfrak{M}^\Gamma} = |\varphi| \cap R^A(\Gamma) \in N^\Gamma(\Delta)$. Thus, $\Delta \in [\![\langle\ ]\varphi]\!]_{\mathfrak{M}^\Gamma}$.

  Suppose that $\Delta \notin |\langle\ ]\varphi|_{\mathbf{EMA}} \cap R^A(\Gamma)$. If $\Delta \notin R^A(\Gamma)$, then obviously $\Delta \notin [\![\Box\varphi]\!]_{\mathfrak{M}^\Gamma}$. So, assume for the remainder of the proof that $\Delta \in R^A(\Gamma)$. Then, $\langle\ ]\varphi \notin \Delta$. We must show that $\mathfrak{M}, \Delta \not\models \Box\varphi$. That is, we must show that for all $X \in N^\Gamma(\Delta)$, $X \neq [\![\varphi]\!]_{\mathfrak{M}^\Gamma}$. This is a consequence of the following claim:

  Claim. For each ψ with $\Box\psi \in \Delta$ there is a maximally consistent set Δ' such that $\Delta' \in |\psi|_{\mathbf{EMA}} \cap R^A(\Gamma)$, but $\varphi \notin \Delta'$.

  Proof of claim. Let $\Delta_0' = \Gamma^A \cup \{\neg\varphi\} \cup \{\psi\}$. First of all, since $\langle\ ]\varphi \notin \Delta$, using the axiom scheme *A-N*, we have $A\varphi \notin \Delta$. Thus, since $\Delta \in R^A(\Gamma)$, $A\varphi \notin \Gamma$ (why?). Therefore, $\varphi \notin \Gamma^A$. We now show that Δ_0' is consistent. Suppose not. Then, $\Gamma^A \cup \{\neg\varphi\} \cup \{\psi\} \vdash_{\mathbf{EMA}} \bot$. Using standard propositional reasoning, $\Gamma^A \cup \{\psi\} \vdash_{\mathbf{EMA}} \varphi$. By Lemma 2.71, $A\Gamma^A \cup \{\Box\psi\} \vdash_{\mathbf{EMA}} \Box\varphi$. Since $A\Gamma^A \subseteq \Gamma^A \subseteq \Delta$ and $\Box\psi \in \Delta$, we have $\Box\varphi \in \Delta$, which is a contradiction. Thus, Δ_0' is consistent. By Lindenbaum's Lemma, there is a maximally consistent set Δ' such that $\Delta_0' \subseteq \Delta'$. Therefore, $\Delta' \in |\psi|_{\mathbf{EMA}} \cap R^A(\Gamma)$ but $\varphi \notin \Delta'$.

- ($[\![A\varphi]\!]_{\mathfrak{M}^\Gamma} = |A\varphi|_{\mathbf{EMA}} \cap R^A(\Gamma)$) Suppose that $\Delta \in |A\varphi|_{\mathbf{EMA}} \cap R^A(\Gamma)$. Then, in particular, $A\varphi \in \Delta$. Let $\Delta' \in R^A(\Gamma)$. Then, since $\Delta \in R^A(\Gamma)$, we have $\Delta\ R^A\ \Delta'$. Hence, since $A\varphi \in \Delta$ and $\Delta^A \subseteq \Delta'$, we have $\varphi \in \Delta'$. Thus $R^A(\Gamma) \subseteq |\varphi|_{\mathbf{EMA}}$. Then, by the induction hypothesis, $[\![\varphi]\!]_{\mathfrak{M}^\Gamma} = |\varphi|_{\mathbf{AMA}} \cap R^A(\Gamma) = R^A(\Gamma)$. Thus, $\Delta \in [\![A\varphi]\!]_{\mathfrak{M}^\Gamma}$.

  Suppose that $\Delta \notin |A\varphi|_{\mathbf{EMA}} \cap R^A(\Gamma)$. Again we can assume that $\Delta \in R^A(\Gamma)$. Then, $A\varphi \notin \Delta$. We must show that there is a $\Delta' \in R^A(\Gamma)$ such that $\varphi \notin \Delta'$. Let $\Delta_0' = \Gamma^A \cup \{\neg\varphi\}$. We claim that Δ_0' is consistent. Suppose not. Then, $\Gamma^A \cup \{\neg\varphi\} \vdash_{\mathbf{EMA}} \bot$. Using standard propositional reasoning, there are formulas $\alpha_1, \ldots, \alpha_n \in \Gamma^A$ such that $\vdash_{\mathbf{EMA}} (\alpha_1 \wedge \cdots \wedge \alpha_n) \to \varphi$. Using *A-Nec* and *A-K*, standard modal reasoning gives us $\vdash_{\mathbf{EMA}} (A\alpha_1 \wedge \cdots \wedge A\alpha_n) \to A\varphi$. Hence, $(A\alpha_1 \wedge \cdots \wedge A\alpha_n) \to A\varphi \in \Delta$. Since, for each $i = 1, \ldots, n$, $A\alpha_i \in \Gamma$, we conclude that $A\alpha_i \in \Delta$ for each $i = 1, \ldots, n$. Thus, $A\varphi \in \Delta$, which is a contradiction. Therefore, Δ_0' is consistent. By Lindenbaum's

Lemma, there is a maximally **EMA**-consistent set Δ' such that $\Delta'_0 \subseteq \Delta'$. Then, $\Delta' \notin |\varphi|_{\mathbf{EMA}} \cap R^A(\Gamma) = [\![\varphi]\!]_{\mathfrak{M}^\Gamma}$. Thus, $\Delta \notin [\![A\varphi]\!]_{\mathfrak{M}^\Gamma}$.

QED

Theorem 2.73 *The logic* **EMA** *is sound and strongly complete with respect to* F_{EMA} *of neighborhood frames.*

2.2.4 General Neighborhood Frames

General frames are an important tool for modal logicians. See Blackburn et al. (2001) for an overview of general frames and their use in the model theory of modal logic. I will not discuss this interesting theory here. Instead I will show how to adapt the definition of a general frame to the neighborhood setting and prove a general completeness theorem.

Definition 2.74 (General Neighborhood Frame) A **general neighborhood frame** is a tuple $\mathfrak{F}^g = \langle W, N, \mathcal{A}\rangle$, where W is a non-empty set of states, N is a neighborhood function, and $\mathcal{A}$ is a collection of subsets of W closed under intersections, complements, and the m_N operator. ◁

We say a valuation $V : \mathsf{At} \to \wp(W)$ is **admissible** for a general frame $\langle W, N, \mathcal{A}\rangle$ provided, for each $p \in \mathsf{At}$, $V(p) \in \mathcal{A}$.

Definition 2.75 (General Neighborhood Model) Suppose that $\mathfrak{F}^g = \langle W, N, \mathcal{A}\rangle$ is a general neighborhood frame. A general neighborhood model based on $\mathfrak{F}^g$ is a tuple $\mathfrak{M}^g = \langle W, N, \mathcal{A}, V\rangle$ where V is an admissible valuation. ◁

Truth for the basic modal logic on general models is defined as usual (cf. Definition 1.15)

Lemma 2.76 *Let* $\mathfrak{M}^g$ *be an general neighborhood model. Then for each* $\varphi \in \mathcal{L}$, $[\![\varphi]\!]_{\mathfrak{M}^g} \in \mathcal{A}$.

Proof. The proof is an easy induction over the structure of φ. QED

Given a modal logic **L** containing **E**, it is easy to show that the set $\mathcal{A}_{\mathbf{L}} = \{|\varphi|_{\mathbf{L}} \mid \varphi \in \mathcal{L}\}$ is a Boolean algebra and closed under the m_N operator. A general frame is called a **L**-general frame, if **L** is valid on that frame. We show that for each modal logic **L** the canonical frame is a **L**-frame.

Theorem 2.77 *Let* **L** *be any logic extending* **E**. *Then, we have*

$$\mathfrak{F}^g_{\mathbf{L}} \models \mathbf{L}.$$

Corollary 2.78 *Every modal logic* **L** *extending* **E** *is sound and strongly complete with respect to a class of general neighborhood frames.*