Neighborhood Semantics for Modal Logic
Lecture 8

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Text

Neighborhood semantics for modal logic (Draft)

Ch 1: Introduction and Motivation

Ch 2: Core Theory: Expressivity, Completeness, Decidability, Complexity, Correspondence Theory

Ch 3: Richer Languages: Fixed-point operators, First-order extensions, Dynamic operators

Schedule

- ✓ Lecture 1: June 1st, 14h00-16h30
- ✓ Lecture 2: June 2nd 12h30-14h30
- ✓ Lecture 3: June 7th, 14h00-16h30
- ✓ Lecture 4: June 8th, 11h00-13h00
- ✓ Lecture 5: June 8th, 14h00-16h30
- ✓ Lecture 6: June 9th, 12h30-14h30
- ✓ Lecture 7: June 13th,12h30-15h00
- Lecture 8: June 14th, 10h00-13h00
- Lecture 9 Presentations (solutions to problems etc.): June 15th, 10h00-13h00

Richer Languages

- ✓ Normal + non-normal modalities
- ✓ First-order extensions
- Dynamics
  - ✓ Dynamics on neighborhoods (updating neighborhood models, evidence dynamics)
  - ✓ Dynamics with neighborhoods (Game logic)
- Fixed-point operators/group notions (group evidence, common belief)

Suppose that $\mathcal{A} = \{1, \ldots, n\}$ is a finite set of agents.

$$p \mid \neg\varphi \mid \varphi \wedge \psi \mid \Box_i \varphi$$

where $p \in \mathsf{At}$ and $i \in \mathcal{A}$.

Multi-Agent Neighborhood Models

$$\langle W, \{N_i\}_{i\in\mathcal{A}}, V\rangle$$

- For each $i \in \mathcal{A}$, $N_i : W \to \wp(\wp(W))$ is a neighborhood function.
- $\mathcal{M}, w \models \Box_i\varphi$ iff $[\![\varphi]\!]_{\mathcal{M}} \in N_i(w)$.

Multi-Agent Evidence Logic

Social notions: Let $\mathcal{M} = \langle W, \mathcal{E}_i, \mathcal{E}_j, V \rangle$ be a multiagent evidence model. What evidence does the group i, j have?

- $\mathcal{M}, w \models \Box^{\{i,j\}}\varphi$ iff there is a $X \in \mathcal{E}_i \cup \mathcal{E}_j$ such that $X \subseteq [\![\varphi]\!]_{\mathcal{M}}$

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- $\mathcal{M}, w \models [i \sqcap j]\varphi$ iff there exists $X \in \mathcal{E}_i \sqcap \mathcal{E}_j$ with $X \subseteq [\![\varphi]\!]_{\mathcal{M}}$
  $\mathcal{E}_i \sqcap \mathcal{E}_j = \{ Y \mid \emptyset \neq Y = X \cap X'$ with $X \in \mathcal{E}_i$ and $X' \in \mathcal{E}_j \}$

Everyone Believes

Suppose that $G \subseteq \mathcal{A}$ is a non-empty set of agents and $\mathcal{M} = \langle W, \{N_i\}_{i \in \mathcal{A}}, V \rangle$ is a multi-agent neighborhood model.

$N_G : W \to \wp(\wp(W))$ be a neighborhood function where for all $w \in W$, $N_G(w) = \bigcap_{i \in G} N_i(w)$.

$$\mathcal{M}, w \models \Box_G \varphi \text{ iff } [\![\varphi]\!]_{\mathcal{M}} \in N_G(w).$$

Everyone Believes

1. If, for all $i \in G$, N_i is monotonic, then N_G is monotonic.
2. If, for all $i \in G$, N_i is augmented, then N_G is augmented.
3. Is there a property P of neighborhood functions such that for each $i \in G$, N_i has property P yet N_G does not have property P?
4. RE plus the axiom

$$\Box_G \varphi \leftrightarrow \bigwedge_{i \in G} \Box_i \varphi$$

is sound and complete for all multi-agent neighborhood frames in which N_G is defined as above.

Distributed Belief

Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are two neighborhoods:

$$\mathcal{X} \sqcap \mathcal{Y} = \{Z \mid Z = X \cap Y, \text{ for some } X \in \mathcal{X} \text{ and } Y \in \mathcal{Y}\}.$$

For each $\emptyset \neq G \subseteq \mathcal{A}$, define the aggregate neighborhood function $N_G^{\sqcap} : W \to \wp(\wp(W))$ as follows: For each $w \in W$,

$$N_G^{\sqcap}(w) = \sqcap_{i \in G} N_i(w).$$

$$\mathcal{M}, w \models [\sqcap]_G \varphi \text{ iff } [\![\varphi]\!]_{\mathcal{M}} \in N_G^{\sqcap}(w).$$

$W = \{w_1, w_2, w_3, w_4\}$

$V(p) = \{w_1, w_3\}$ and $V(q) = \{w_1, w_2\}$.

- $N_1(w_1) = \{\{w_1, w_3\}, \{w_1, w_2, w_3\}, \{w_1, w_3, w_4\}, W\}$, and
- $N_2(w_1) = \{\{w_1, w_4\}, \{w_1, w_2, w_4\}, \{w_1, w_3, w_4\}, W\}$.

$\mathcal{M}, w_1 \models \Box_1 p$ and $\mathcal{M}, w_1 \models \Box_2(p \rightarrow q)$.
$\mathcal{M}, w_1 \not\models \Box_1 q$ and $\mathcal{M}, w_1 \not\models \Box_2 q$.

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$\mathcal{M}, w_1 \models [\sqcap]_{\{1,2\}} q$.

$$
\begin{array}{rcl}
N_1(w_1) \sqcap N_2(w_1) & = & \{\{w_1\}, \{w_1, w_2\}, \{w_1, w_3\}, \\
 & & \{w_1, w_4\}, \{w_1, w_2, w_3\}, \{w_1, w_3, w_4\}, \\
 & & \{w_1, w_2, w_4\}, W\}.
\end{array}
$$

Note: $N_G^{\sqcap}(w)$ may contain $\emptyset$ even when $\emptyset \notin N_i(w)$ for each $i \in G$.

1. In a multi-agent neighborhood models in which each neighborhood contains the unit (for all $i \in \mathcal{A}$, for all $w \in W$, $W \in N_i(w)$), the formula $\bigwedge_{i \in G}(\Box_i \varphi \to [\sqcap]_G \varphi)$ is valid.
2. Suppose that for all $i \in G$, N_i is augmented. Prove that $N_G^{\sqcap}$ is augmented, and that $R_{N_G^{\sqcap}} = \bigcap_{i \in G} R_{N_i}$.

Common Knowledge/Belief

"*Common Knowledge*" is informally described as what any fool would know, given a certain situation: It encompasses what is relevant, agreed upon, established by precedent, assumed, being attended to, salient, or in the conversational record.

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It is not Common Knowledge who "defined" Common Knowledge!

The first formal definition of common knowledge?

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Fixed-point definition: $\gamma :=$ *i* and *j* know that (φ and γ)

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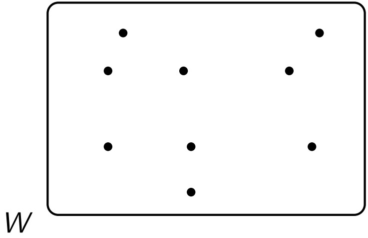
Shared situation: There is a *shared situation* s such that (1) s entails φ, (2) s entails everyone knows φ, plus other conditions

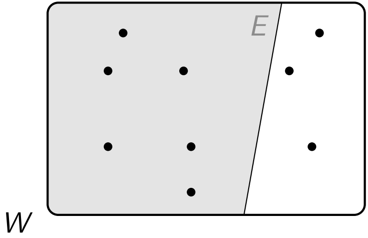
H. Clark and C. Marshall. *Definite Reference and Mutual Knowledge*. 1981.

M. Gilbert. *On Social Facts*. Princeton University Press (1989).

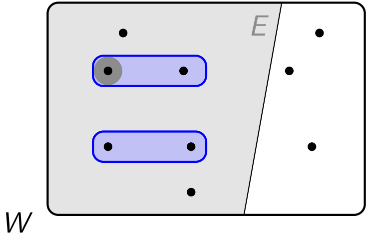
P. Vanderschraaf and G. Sillari. *"Common Knowledge", The Stanford Encyclopedia of Philosophy (2009)*.
http://plato.stanford.edu/entries/common-knowledge/.



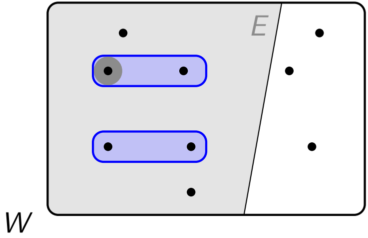
W is a set of **states** or **worlds**.



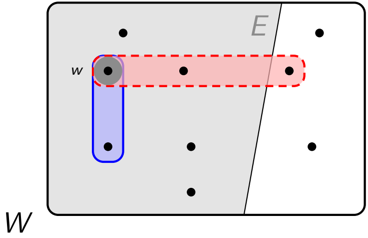
An **event/proposition** is any (definable) subset $E \subseteq W$



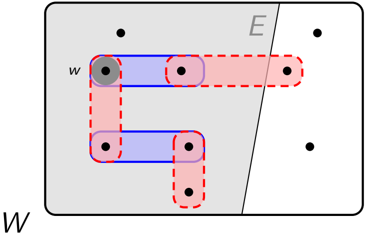
The agents receive signals in each state. States are considered equivalent for the agent if they receive the same signal in both states.



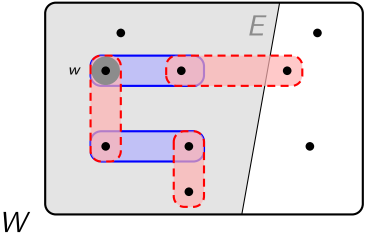
Knowledge Function: $K_i : \wp(W) \to \wp(W)$ where $K_i(E) = \{w \mid R_i(w) \subseteq E\}$



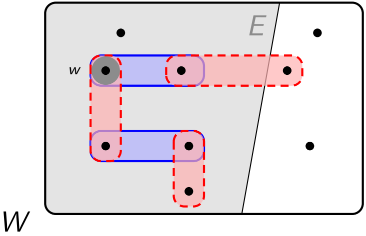
$w \in K_A(E)$ and $w \notin K_B(E)$



The model also describes the agents' **higher-order knowledge/beliefs**

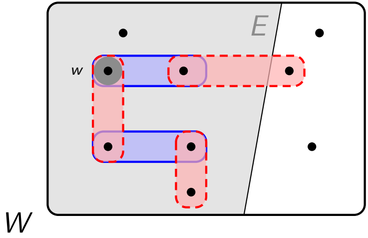


Everyone Knows: $K(E) = \bigcap_{i\in\mathcal{A}} K_i(E)$, $K^0(E) = E$, $K^m(E) = K(K^{m-1}(E))$

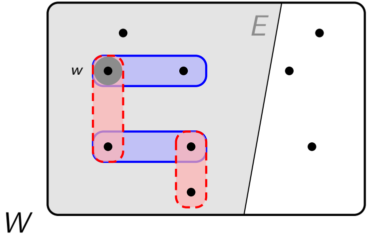


Common Knowledge: $C : \wp(W) \rightarrow \wp(W)$ with

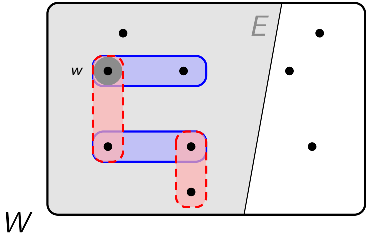
$$C(E) = \bigcap_{m \geq 0} K^m(E)$$



$w \in K(E)$ $w \notin C(E)$



$w \in C(E)$



Fact. $w \in C(E)$ if every finite path starting at w ends in a state in E

An Example

Two players Ann and Bob are told that the following will happen. Some positive integer n will be chosen and *one* of n, $n+1$ will be written on Ann's forehead, the other on Bob's. Each will be able to see the other's forehead, but not his/her own.

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Suppose the number are (2,3).

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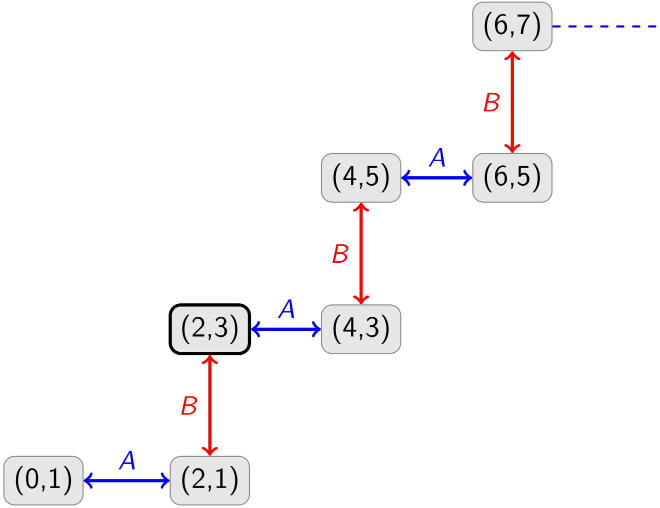
Suppose the number are (2,3).

Do the agents know there numbers are less than 1000?

Is it common knowledge that their numbers are less than 1000?


(6,7)
B
(4,5)
A
(6,5)
B
(2,3)
A
(4,3)
B
(0,1)
A
(2,1)

Fact. For all $i \in \mathcal{A}$ and $E \subseteq W$, $K_i C(E) = C(E)$.

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Suppose you are told "Ann and Bob are going together,"' and respond "sure, that's common knowledge." What you mean is not only that everyone knows this, but also that the announcement is pointless, occasions no surprise, reveals nothing new; in effect, that the situation after the announcement does not differ from that before. ...the event "Ann and Bob are going together" — call it E — is common knowledge if and only if some event — call it F — happened that entails E and also entails all players' knowing F (like all players met Ann and Bob at an intimate party). *(Aumann, pg. 271, footnote 8)*

Fact. For all $i \in \mathcal{A}$ and $E \subseteq W$, $K_i C(E) = C(E)$.

An event F is **self-evident** if $K_i(F) = F$ for all $i \in \mathcal{A}$.

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The following axiomatize common knowledge:

- $C(\varphi \to \psi) \to (C\varphi \to C\psi)$
- $C\varphi \to (\varphi \wedge EC\varphi)$ (Fixed-Point)
- $C(\varphi \to E\varphi) \to (\varphi \to C\varphi)$ (Induction)

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$$f_E(X) = K(E \cap X) = \bigcap_{i \in \mathcal{A}} K_i(E \cap X)$$

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- f_E is monotonic:

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- (Tarski) Every monotone operator has a greatest (and least) fixed point
- Let $K^*(E)$ be the greatest fixed point of f_E.
- **Fact**. $K^*(E) = C(E)$.

The Fixed-Point Definition

Separating the fixed-point/iteration definition of common knowledge/belief:

J. Barwise. *Three views of Common Knowledge*. TARK (1987).

J. van Benthem and D. Saraenac. *The Geometry of Knowledge*. Aspects of Universal Logic (2004).

A. Heifetz. *Iterative and Fixed Point Common Belief*. Journal of Philosophical Logic (1999).

Common r-belief

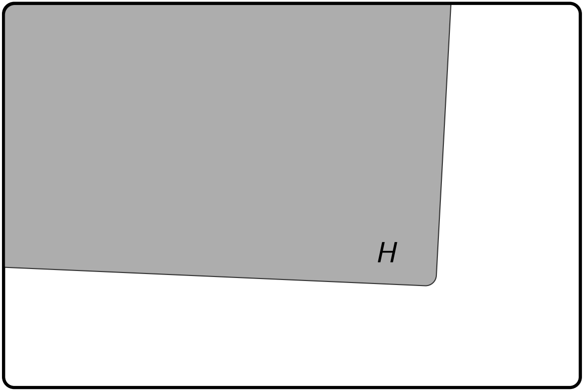
The typical example of an event that creates common knowledge is a **public announcement**.

Common r-belief

The typical example of an event that creates common knowledge is a **public announcement**.

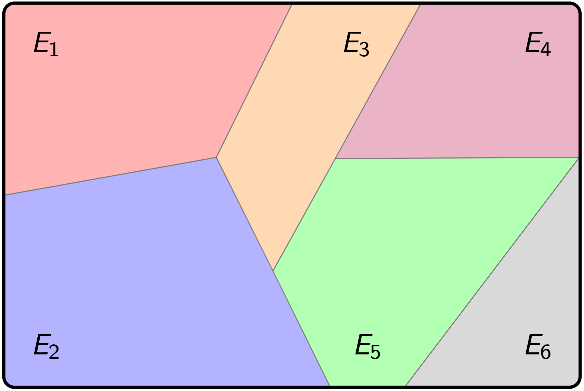
Shouldn't one always allow for some small probability that a participant was absentminded, not listening, sending a text, checking Facebook, proving a theorem, asleep, ...

D. Monderer and D. Samet. *Approximating Common Knowledge with Common Beliefs*. Games and Economic Behavior (1989).

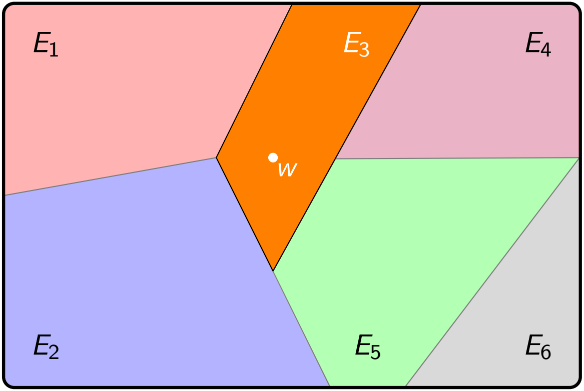


An **event/proposition** is a (definable) subset $H \subseteq W$.

A σ-**algebra** is the collection of events/propositions (closed under countable unions and complementation)



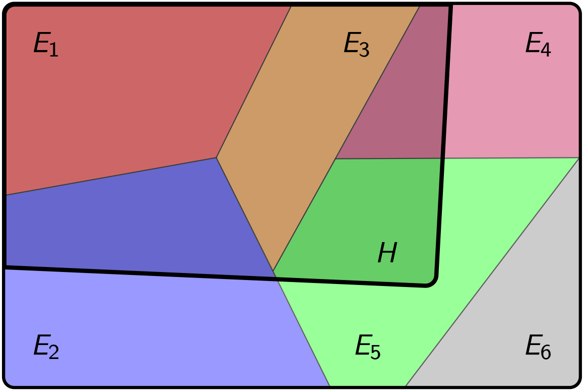
An **experiment/question/set of signals** is a partition $\mathcal{E}$ on W.



An **experiment/question/set of signals** is a partition $\mathcal{E}$ on W.

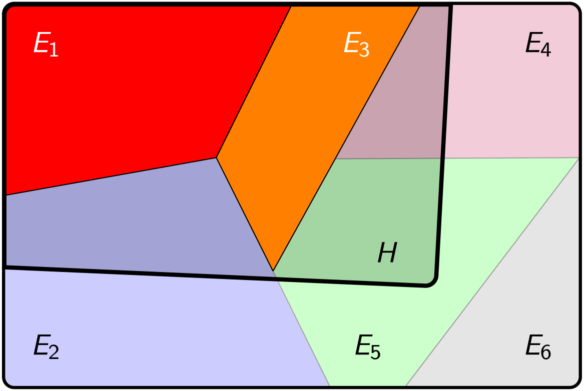
If $w \in W$, let $\mathcal{E}[w] = E$ where $w \in E \in \mathcal{E}$.

E.g, if $\mathcal{E} = \{E_1, E_2, E_3, E_4, E_5, E_6\}$, then $\mathcal{E}[w] = E_3$

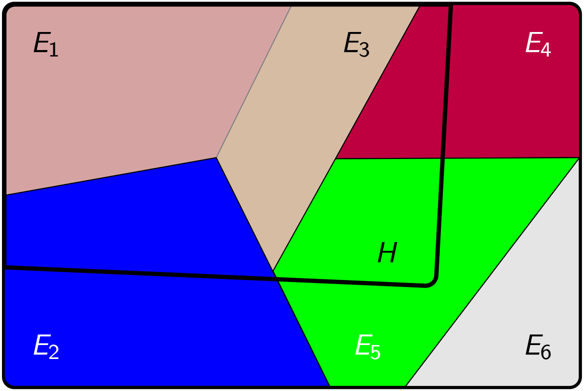


$K_{\mathcal{E}} : \wp(W) \to \wp(W)$, where for $H \subseteq W$,

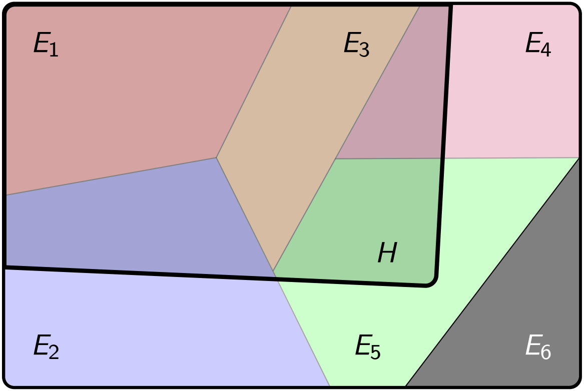
$$K_{\mathcal{E}}(H) = \{w \mid \mathcal{E}[w] \subseteq H\}$$



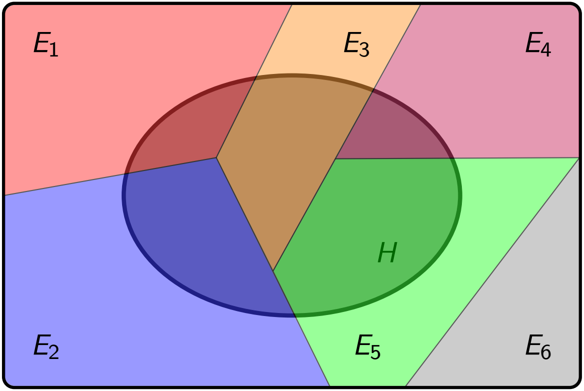
$$K_{\mathcal{E}}(H) \quad = \quad E_1 \cup E_3$$



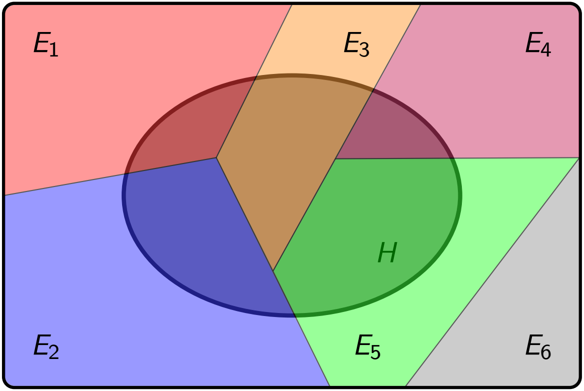
$$
\begin{aligned}
K_{\mathcal{E}}(H) &= E_1 \cup E_3 \\
-K_{\mathcal{E}}(H) \cap -K_{\mathcal{E}}(-H) &= E_2 \cup E_4 \cup E_5
\end{aligned}
$$



$$
\begin{aligned}
K_{\mathcal{E}}(H) &= E_1 \cup E_3 \\
-K_{\mathcal{E}}(H) \cap -K_{\mathcal{E}}(-H) &= E_2 \cup E_4 \cup E_5 \\
K_{\mathcal{E}}(-H) &= E_6
\end{aligned}
$$

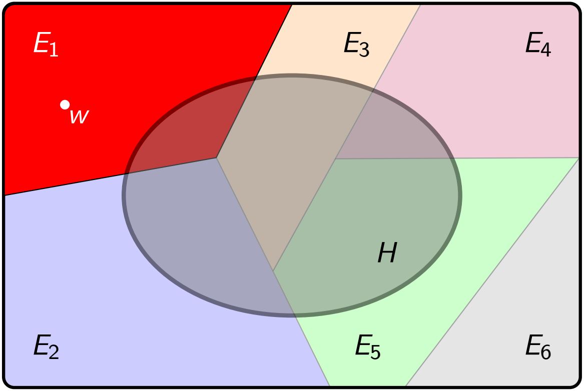


If p is a probability on W (with respect to a σ-algebra $\mathcal{F}$)



If p is a probability on W (with respect to a σ-algebra $\mathcal{F}$)

The posterior at w with respect to $\mathcal{E}$ is $p_{\mathcal{E},w}(H) = p(H \mid \mathcal{E}[w])$

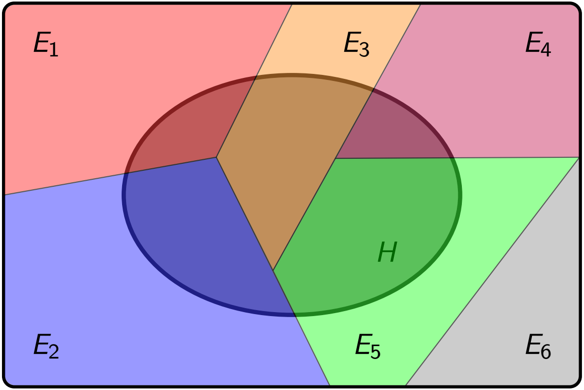


If p is a probability on W (with respect to a σ-algebra $\mathcal{F}$)

The posterior at w with respect to $\mathcal{E}$ is $p_{\mathcal{E},w}(H) = p(H \mid \mathcal{E}[w])$

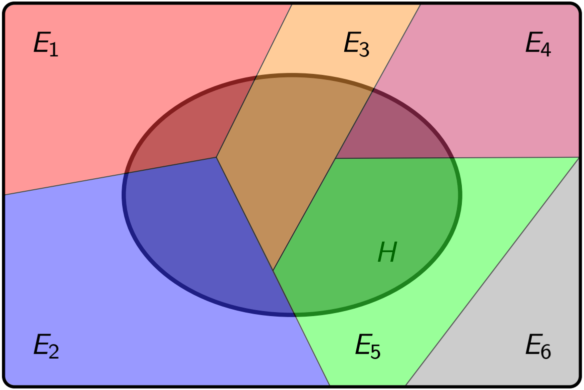
E.g., $p_{\mathcal{E},w}(H) = p(H \mid E_1)$

From Knowledge to r-Belief



Given a partition $\mathcal{E}$, define $K_{\mathcal{E}} : \wp(W) \to \wp(W)$ as:
$K_{\mathcal{E}}(H) = \{w \mid \mathcal{E}[w] \subseteq H\}$

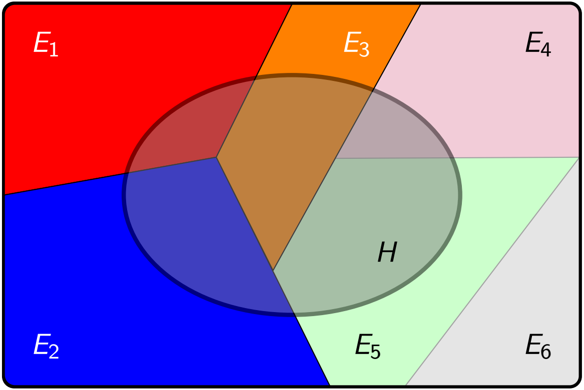
From Knowledge to r-Belief



Given $r \in [0, 1]$ and a partition $\mathcal{E}$, define $B^r_{\mathcal{E}} : \wp(W) \rightarrow \wp(W)$ as:

$$B^r_{\mathcal{E}}(H) = \{w \mid p_{\mathcal{E},w}(H) \geq r\}$$

From Knowledge to r-Belief



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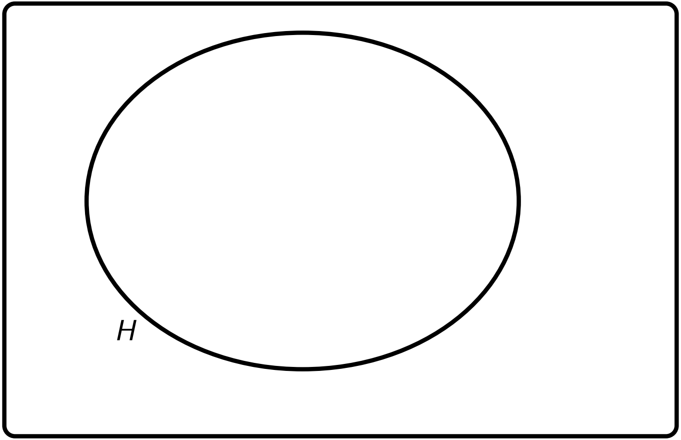
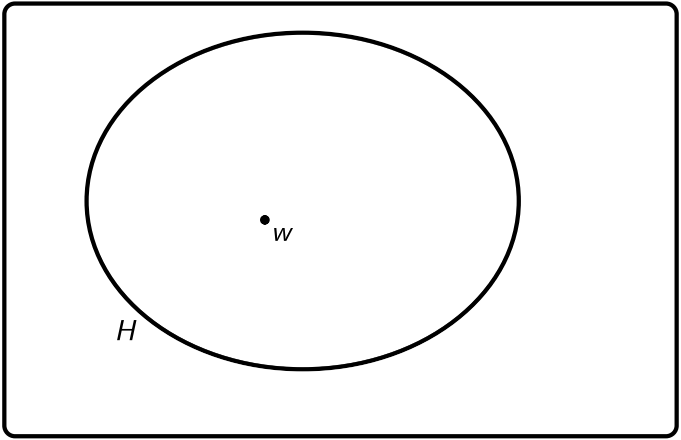
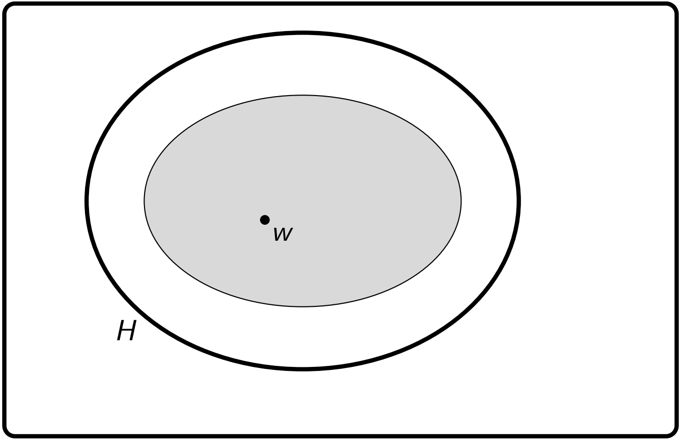
From Common Knowledge to Common r-Belief

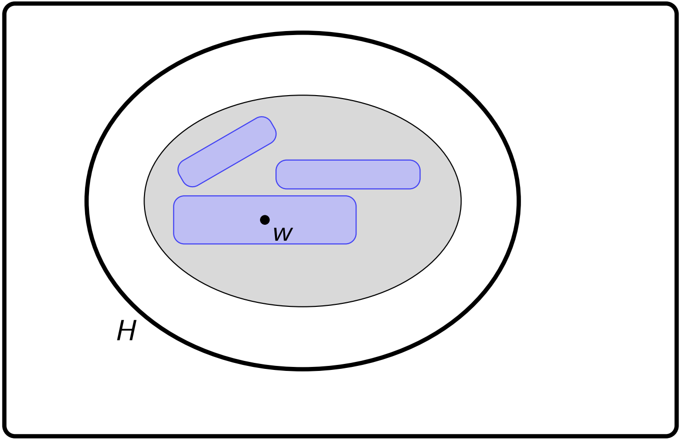
Suppose that $C : \wp(W) \rightarrow \wp(W)$ is a common knowledge operator. TFAE

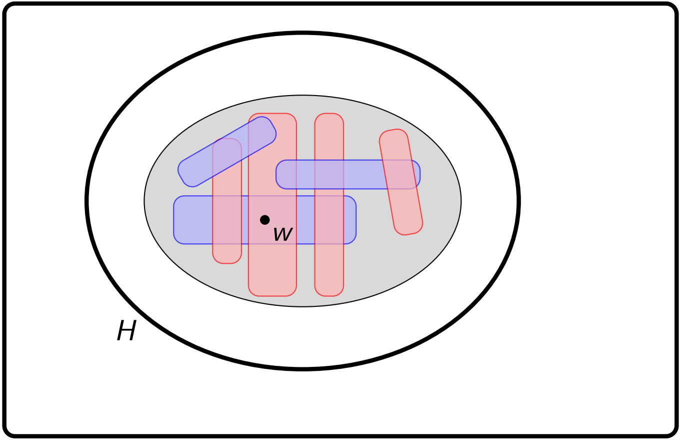
1. $w \in C(H) = \bigcap_{m \geq 0} K^m(H)$
2. $I_c(w) \subseteq H$
3. There is a set $F \subseteq W$ such that
    3.1 $w \in F \subseteq K(F) = \bigcap_i K_i(F)$
    3.2 $F \subseteq H$


H


w
H


w
H


w
H


w
H

From Common Knowledge to Common r-Belief

$$B_i^r(E) = \{w \mid p(E \mid \mathcal{E}_i[w]) \geq r\}$$

From Common Knowledge to Common r-Belief

$B_i^r(E) = \{w \mid p(E \mid \mathcal{E}_i[w]) \geq r\}$

F is an **evident r-belief** if for each $i \in \mathcal{A}$, $F \subseteq B_i^r(F)$

From Common Knowledge to Common r-Belief

$B_i^r(E) = \{w \mid p(E \mid \mathcal{E}_i[w]) \geq r\}$

F is an **evident** r**-belief** if for each $i \in \mathcal{A}$, $F \subseteq B_i^r(F)$

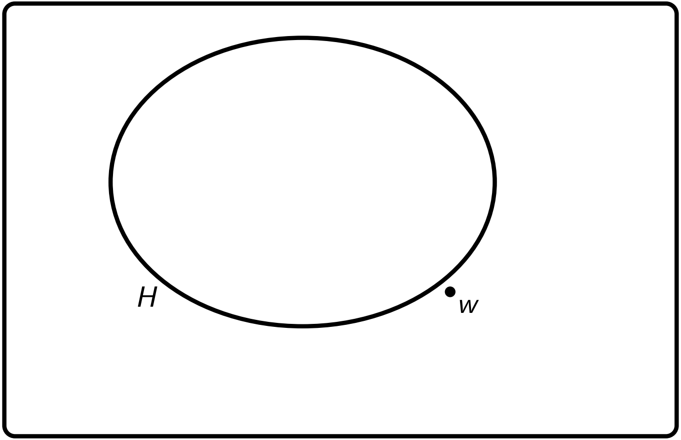
An event H is **common** r**-belief** at w if there exists and evident r-belief event F such that $w \in F$ and for all $i \in \mathcal{A}$, $F \subseteq B_i^r(H)$

$w \in C(H)$ iff there is an event $F \subseteq W$ such that

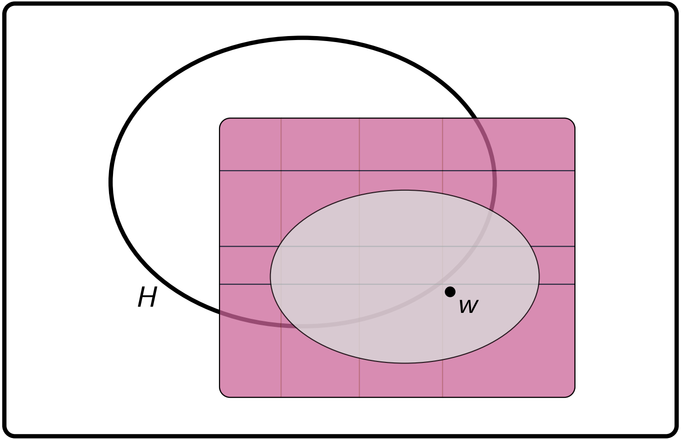
1. $w \in F \subseteq K(F) = \bigcap_i K_i(F)$
2. $F \subseteq H$

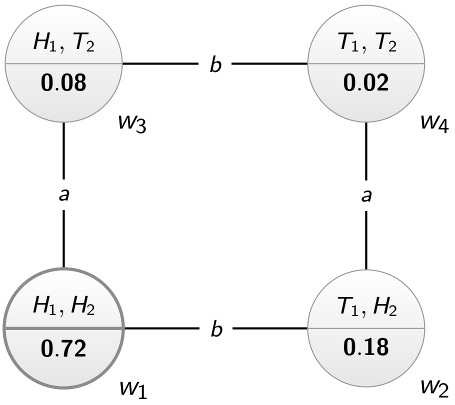
$w \in C^r(H)$ iff there is an event $F \subseteq W$ such that

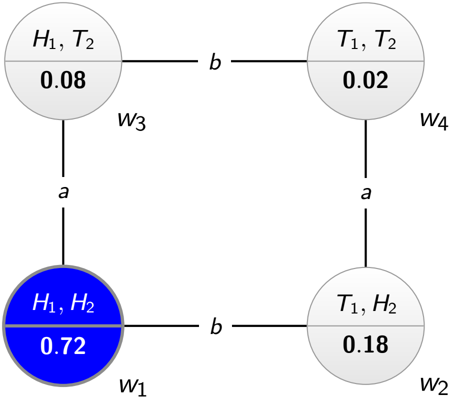
1. $w \in F \subseteq B^r(F) = \bigcap_i B_i^r(F)$
2. $F \subseteq B^r(H)$



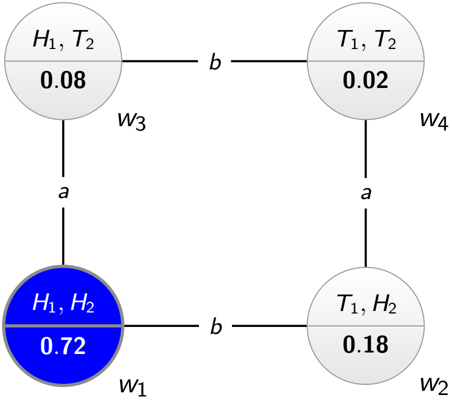
H
w


H
w


H_1, T_2
0.08
w_3
b
T_1, T_2
0.02
w_4
a
a
H_1, H_2
0.72
w_1
b
T_1, H_2
0.18
w_2



- $\{w_1\} = B_a^{0.9}(H_1 \cap H_2) \cap B_b^{0.8}(H_1 \cap H_2)$.
- $X = \{w_1\}$ is an evident 0.8-belief for both Ann and Bob.



- $\{w_1\} = B_a^{0.9}(H_1 \cap H_2) \cap B_b^{0.8}(H_1 \cap H_2)$.
- $X = \{w_1\}$ is an evident 0.8-belief for both Ann and Bob.
- $X \subseteq B_a^{0.8}(H_1 \cap H_2) \cap B_b^{0.8}(H_1 \cap H_2)$.
- $w_1 \in C_{a,b}^{0.8}(H_1 \cap H_2)$.

Common Belief on Neighborhood Structures

Suppose that $\mathcal{M} = \langle W, \{N_i\}_{i\in\mathcal{A}}, V\rangle$ is a multi-agent neighborhood model.

A set $X \subseteq W$ is i-**evident** provided $X \subseteq \{w \mid X \in N_i(w)\}$.

A set X is G-**evident** if it is i-evident for all $i \in G$. To simplify the notation, we say that X is **evident** if it is $\mathcal{A}$-evident.

Common Belief, version 1 Suppose that $\mathcal{M} = \langle W, \{N_i\}_{i\in\mathcal{A}}, V\rangle$ is a monotonic multi-agent neighborhood model. For each $G \subseteq \mathcal{A}$, define a function $C^1_G : \wp(W) \to \wp(W)$ as follows:

$$C^1_G(Y) = \{w \mid \exists\ G\text{-evident set } X \text{ s.th. } w \in X \text{ and } X \subseteq m_G(Y)\}.$$

$$C^1_G(Y) = \bigcup\{X \mid X \subseteq m_G(X) \cap m_G(Y)\}.$$

$$\mathcal{M}, w \models \Box^*_G\varphi \text{ iff } w \in C^1_G([\![\varphi]\!]_\mathcal{M}).$$

Let $\mathbf{EM_n^C}$ be the smallest set of formulas that contains all tautologies in the above language, contains all instances of the following axiom schemes and closed under the following rules:

$$(RM_n) \qquad \frac{\varphi \rightarrow \psi}{\Box_i\varphi \rightarrow \Box_i\psi}$$

$$(\text{CB}) \qquad \Box_G^*\varphi \rightarrow \Box_G\varphi$$

$$(\text{FP1}) \qquad \Box_G^*\varphi \rightarrow \Box_G\Box_G^*\varphi$$

$$(\text{FP2}) \qquad \frac{(\varphi \rightarrow \Box_G\varphi) \wedge (\varphi \rightarrow \Box_G\psi)}{\varphi \rightarrow \Box_G^*\psi}$$

If $\vdash_{\mathbf{EM_n^C}} \varphi \rightarrow \psi$, then $\vdash_{\mathbf{EM_n^C}} \Box_G^* \varphi \rightarrow \Box_G^* \psi$.

$\Box_G^0\varphi = \Box_G\varphi$ and for $k > 0$, for each $k > 0$, $\Box_G^k\varphi = \Box_G(\Box_G^{k-1}\varphi)$.

$k \in \mathbb{N}$, $\Box_G^*\varphi \to \Box_G^k$ is valid.

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$k \in \mathbb{N}$, $\Box_G^*\varphi \to \Box_G^k$ is valid.

The key observation is that if $w \in C_G^1(\llbracket\varphi\rrbracket_{\mathcal{M}})$, then there is a G-evident set X such that $w \in X \subseteq m_G(X) \cap m_G(\llbracket\varphi\rrbracket_{\mathcal{M}})$. By monotonicity, we have that $m_G(X) \subseteq m_G(m_G(\llbracket\varphi\rrbracket_{\mathcal{M}}))$. Since X is G-evident, we have that $w \in X \subseteq m_G(X) \subseteq m_G(m_G(\llbracket\varphi\rrbracket_{\mathcal{M}}))$.

- $\vdash_{\mathbf{EM_n^C}} \Box_G^*\varphi \to \Box_G^k\varphi$ for all $k \in \mathbb{N}$.

The truth of $\Box_G^* \varphi$ implies that each of the formulas in the set $\{\Box_G^k \varphi \mid k \in \mathbb{N}\}$ are true.

The truth of $\Box^*_G\varphi$ implies that each of the formulas in the set $\{\Box^k_G\varphi \mid k \in \mathbb{N}\}$ are true.

Can conclude that $\Box^*_G\varphi$ true if we know that conjunctions of formulas from $\{\Box^k_G\varphi \mid k \in \mathbb{N}\}$ are true?

There is more than one way to form intersections from the set $\{[\![\Box_G^k \varphi]\!]_{\mathcal{M}} \mid k \in \mathbb{N}\}$. Since the neighborhoods are not closed under intersections, it is not hard to find a set X such that:

$$m_G(X \cap m_G(X)) \neq m_G(X) \cap m_G(m_G(X)).$$

For a set $X \subseteq W$, define X_α, where α is any infinite cardinal, by transfinite induction:

$$\begin{array}{rcl} X_0 & = & m_G(X) \\ X_\alpha & = & \bigcap_{\beta<\alpha} X_\beta \cap m_G(\bigcap_{\beta<\alpha} X_\beta) \end{array}$$

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Note that the above sequence of sets is decreasing. Thus, for any set W, there must be some μ such that $X_\mu = X_{\mu+1}$.

For any set W, there must be some μ such that $X_\mu = X_{\mu+1}$. This brings us to the second issue.

In general, μ may not be $\aleph_0$ (the first countable cardinal). That is, $\bigcap_{n\in\mathbb{N}} X_n$ may contains states that are not in $m_G(\bigcap_{n\in\mathbb{N}} X_n)$.

Fact. Suppose that $\mathcal{M} = \langle W, \{N_i\}_{i\in\mathcal{A}}, V\rangle$ is a multi-agent neighborhood model where each N_i are augmented. For any $X \subseteq W$, prove that X_α stabilizes when $\alpha = \aleph_0$.

Proposition. Suppose that $\mathcal{M} = \langle W, \{N_i\}_{i \in \mathcal{A}}, V \rangle$ is a monotonic multi-agent neighborhood model. For any set $Y \subseteq W$ and set $G \subseteq \mathcal{A}$, $C^1_G(Y) = Y_\mu$, where μ is the least cardinal such that $Y_\mu = Y_{\mu+1}$.

Common Belief, version 2 Suppose that $\mathcal{M} = \langle W, \{N_i\}_{i\in\mathcal{A}}, V\rangle$ is a monotonic multi-agent neighborhood model. For each $G \subseteq \mathcal{A}$, define a function $C^2_G : \wp(W) \to \wp(W)$ as follows:

$$C^2_G(Y) = \{w \mid \exists\ G\text{-evident set } X \text{ s.th. } w \in m_G(X) \text{ and } X \subseteq Y\}$$

For a set $X \subseteq W$, define $\hat{X}_\alpha$, where α is any infinite cardinal, by transfinite induction:

$$\begin{aligned} \hat{X}_0 &= m_G(X) \\ \hat{X}_\alpha &= m_G(X \cap \textstyle\bigcap_{\beta<\alpha} \hat{X}_\beta) \end{aligned}$$

Suppose that $\mathcal{M} = \langle W, \{N_i\}_{i \in \mathcal{A}}, V \rangle$ is a monotonic multi-agent neighborhood model. Let $\Box^\star_G$ be a modal operator with the truth clause $\mathcal{M}, w \models \Box^\star_G \varphi$ iff $w \in C^2_G(\llbracket \varphi \rrbracket_\mathcal{M})$.

1. For any set $Y \subseteq W$ and set $G \subseteq \mathcal{A}$, $C^2_G(Y) = \hat{Y}_\mu$, where μ is the least cardinal such that $\hat{Y}_\mu = \hat{Y}_{\mu+1}$.
2. $\Box^\star_G \varphi \rightarrow \Box_G \varphi$ and $\Box^\star_G \varphi \rightarrow \Box_G \Box^\star_G \varphi$ are valid.

Proposition. Suppose that $\mathcal{M} = \langle W, \{N_i\}_{i \in \mathcal{A}}, V \rangle$ is a monotonic multi-agent neighborhood model. For any set $Y \subseteq W$, $C^2_G(Y) \subseteq C^1_G(Y)$.

The converse of the Proposition is not true.
Suppose that $W = \{w, x, y\}$ and all agents have the same neighborhood function $N : W \to \wp(\wp(W))$:

- $N(w) = \{\{w\}, \{v\}, \{x\}, \{w, v\}, \{w, x\}, \{v, x\}, \{w, v, x\}\}$
- $N(v) = \{\{w, v\}, \{w, x\}, \{v, x\}, \{w, v, x\}\}$
- $N(x) = \{\{w, v, x\}\}$

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- $N(x) = \{\{w, v, x\}\}$

Then, $X = \{w\}$ is evident: $m_{\mathcal{A}}(\{w\}) = \{w\}$. This implies that $w \in C^1_{\mathcal{A}}(\{v, x\})$ since $w \in X$, X is evident and $X \subseteq m_{\mathcal{A}}(\{v, x\}) = \{w, v\}$. However, $w \notin C^2_{\mathcal{A}}(\{v, x\})$.

The converse of the Proposition is not true.
Suppose that $W = \{w, x, y\}$ and all agents have the same neighborhood function $N : W \to \wp(\wp(W))$:

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Then, $X = \{w\}$ is evident: $m_{\mathcal{A}}(\{w\}) = \{w\}$. This implies that $w \in C^1_{\mathcal{A}}(\{v, x\})$ since $w \in X$, X is evident and $X \subseteq m_{\mathcal{A}}(\{v, x\}) = \{w, v\}$. However, $w \notin C^2_{\mathcal{A}}(\{v, x\})$.

To see this, note that there are no subsets of $\{v, x\}$ that are evident ($m_{\mathcal{A}}(\{v\}) = m_{\mathcal{A}}(\{x\}) = \{w\}$, $m_{\mathcal{A}}(\{v, x\}) = \{w, v\}$).

L. Lismont, L. and P. Mongin. *Strong completeness theorems for weak logics of common belief*. Journal of Philosophical Logic 32(2), 115 137, 2003.

$B_i\varphi$: i believes that φ

$E\varphi$: Everyone believes that φ

$C\varphi$: There is common belief in φ.

$$f : \wp(W) \to \wp(W)$$

Monotonicity $X \subseteq Y$ implies $f(X) \subseteq f(Y)$

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Reflective-Monotonicity $X \subseteq f(X)$ implies $f(X) \subseteq f(f(X)$

Proposition. If f is quasi-monotonic then f has a greatest fixed-point

$$F = \bigcup \{X \subseteq W \mid X \subseteq f(X)\}$$

LM Common Belief X is common belief at w if there is an X which is belief-closed, believed at w and included in X.

$$F_P = \bigcup\{B_G(X) \mid X \subseteq B_G(X) \cap P\}$$

$$F_P' = \bigcup\{X' \mid X' \subseteq B_G(X \cap P)\}$$

$f_P : \wp(W) \rightarrow \wp(W)$ where $f_P(X) = B_G(X \cap P)$.

MS Common Belief X is common belief at w if there is an Y such that Y is belief-closed, $w \in Y$, and $Y \subseteq B_G(X)$.

$$H_P = \bigcup\{X \mid X \subseteq B_G(X) \cap B_G(P)\}$$

$h_P : \wp(W) \to \wp(W)$ where $h_P(X) = B_G(X) \cap B_G(P)$.

(RE) $$\frac{\varphi \leftrightarrow \psi}{\Box_i\varphi \leftrightarrow \Box_i\psi}$$

(DefE) $$E\varphi \leftrightarrow \bigwedge_i B_i\varphi$$

(RQM) $$\frac{\varphi \rightarrow (E\varphi \wedge \psi)}{E\varphi \rightarrow E\psi}$$

$$(PF_f) \qquad C_f\varphi \rightarrow E(C_f\varphi \wedge \varphi)$$

$$(RI_f) \qquad \frac{\chi \rightarrow E(\chi \wedge \varphi)}{\chi \rightarrow C_f\varphi}$$

$$(PF_h) \qquad C_h\varphi \rightarrow (EC_h\varphi \wedge E\varphi)$$

$$(RI_h) \qquad \frac{\chi \rightarrow (E\chi \wedge E\varphi)}{\chi\varphi \rightarrow C_h\varphi}$$

Theorem. $QMCB_f$ and $QMCB_h$ are sound and **strongly complete**.

Lemma 1. For any W, suppose that $\mathfrak{B}(W) \subseteq \mathfrak{P}(W)$ and $f : \mathfrak{B}(W) \to \mathfrak{B}(W)$ is quasi-monotonic. Then, the following map $f^* : \mathfrak{P}(W) \to \mathfrak{P}(W)$

$$\begin{array}{rcl} f^*(P) & = & f(P) \text{ for } P \in \mathfrak{B}(W) \\ f^*(P) & = & \bigcup\{f(P') \mid P' \in \mathfrak{B}(W), P' \subseteq f(P') \cap P\} \end{array}$$

is quasi-monotonic.

Lemma 2. For any W, suppose that $\mathfrak{B}(W) \subseteq \mathfrak{P}(W)$, $f_0 : \mathfrak{B}(W) \to \mathfrak{B}(W)$ and $f : \mathfrak{B}(W) \to \mathfrak{B}(W)$, f is quasi-monotonic, and for all $X \in \mathfrak{B}(W)$,

$$X \subseteq f_0(X) \Longrightarrow X \subseteq f(X)$$

Take any mapping f^* that coincides with f on $\mathfrak{B}(W)$ and otherwise satisfies any of the following two clauses:

1. $f^*(X) = f(X')$ for some $X' \in \mathfrak{B}(W)$ such that $X' \subseteq X$
2. $f^*(X) = \bigcup\{f(X') \mid X' \in \mathfrak{B}(W), X' \subseteq f(X') \cap X\}$

Then:

$$\bigcup\{X \in \wp(W) \mid X \subseteq f^*(X)\} = \bigcup\{X \in \mathfrak{B}(W) \mid X \subseteq f(X)\}$$

$B_i(X) = |B_i\varphi|$ if $X = |\varphi|$

$B_i(X) = \bigcup\{|E\varphi| \mid |\varphi| \subseteq |E\varphi| \cap X\}$, otherwise.

$B_i(X) = |B_i\varphi|$ if $X = |\varphi|$

$B_i(X) = \bigcup\{|E\varphi| \mid |\varphi| \subseteq |E\varphi| \cap X\}$, otherwise.

Define $B_E(X) = \bigcap_i B_i(X)$

Then,

$B_E(X) = |E\varphi|$ if $X = |\varphi|$

$B_E(X) = \bigcup\{|E\varphi| \mid |\varphi| \subseteq |E\varphi| \cap X\}$, otherwise.

Claim. B_E is quasi-monotonic. (Lemma 1).

Truth Lemma For all φ, $[\![\varphi]\!]_{\mathcal{M}^C} = |\varphi|$.

Claim. $[\![C\psi]\!]_{\mathcal{M}^C} = |C\psi|$.

1. $|C\psi| = \bigcup\{|\chi| \mid |\chi| \subseteq f(|\chi|)\}$
2. $\bigcup\{|\chi| \mid |\chi| \subseteq f(|\chi|)\} = \bigcup\{X \subseteq W^{MC} \mid X \subseteq f(X)\}$

- $f_{|\psi|} = B_E(X \cap |\psi|)$
- $h_{|\psi|} = B_E(X) \cap B_E(|\psi|)$

$$
\begin{aligned}
[\![C\psi]\!]_{\mathcal{M}^C} &= \bigcup\{X \subseteq W^{MC} \mid X \subseteq B_E(X \cap [\![\psi]\!]_{\mathcal{M}^C})\} \\
&= \bigcup\{X \subseteq W^{MC} \mid X \subseteq B_E(X \cap |\psi|)\} \\
&= \bigcup\{X \subseteq W^{MC} \mid X \subseteq f_{|\psi|}(|\chi|)\} \\
&= \bigcup\{|\chi| \mid |\chi| \subseteq f_{|\psi|}(|\chi|)\} \\
&= |C_f\psi|
\end{aligned}
$$

Lecture 9 Presentations (solutions to problems etc.): June 15th, 10:00-12:00

Papers/Solutions to problems due on **July 25** (via email).