Neighborhood Semantics for Modal Logic
Lecture 7

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Text

Neighborhood semantics for modal logic (Draft)

Ch 1: Introduction and Motivation

Ch 2: Core Theory: Expressivity, Completeness, Decidability, Complexity, Correspondence Theory

Ch 3: Richer Languages: Fixed-point operators, First-order extensions, Dynamic operators

Schedule

- ✓ Lecture 1: June 1st, 14h00-16h30
- ✓ Lecture 2: June 2nd 12h30-14h30
- ✓ Lecture 3: June 7th, 14h00-16h30
- ✓ Lecture 4: June 8th, 11h00-13h00
- ✓ Lecture 5: June 8th, 14h00-16h30
- ✓ Lecture 6: June 9th, 12h30-14h30
- Lecture 7: June 13th,12h30-15h00
- Lecture 8: June 14th, 10h00-13h00
- Lecture 9 Presentations (solutions to problems etc.): June 15th, 10h00-13h00

Richer Languages

$$p \mid \neg\varphi \mid \varphi \wedge \psi \mid \Box\varphi \mid \langle\ ]\varphi$$

- $\mathcal{M}, w \models \Box\varphi$ iff $[\![\varphi]\!]_{\mathcal{M}} \in N(W)$
- $\mathcal{M}, w \models \langle\ ]\varphi$ iff there is a $X \in N(W)$ such that $X \subseteq [\![\varphi]\!]_{\mathcal{M}}$

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- $\mathcal{M}, w \models \langle\ ]^{\psi}\varphi$ there is a $X \in N(W)$ such that $X \cap [\![\psi]\!]_{\mathcal{M}} \neq \emptyset$ and $X \cap [\![\psi]\!]_{\mathcal{M}} \subseteq [\![\varphi]\!]_{\mathcal{M}}$.

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- $\mathcal{M}, w \models \langle\,]^{\psi}\varphi$ there is a $X \in N(W)$ such that $X \cap [\![\psi]\!]_{\mathcal{M}} \neq \emptyset$ and $X \cap [\![\psi]\!]_{\mathcal{M}} \subseteq [\![\varphi]\!]_{\mathcal{M}}$.
- $\mathcal{M}, w \models [B]\varphi$ iff for all max-f.i.p. $\mathcal{X} \subseteq N(w)$, $\bigcap \mathcal{X} \subseteq [\![\varphi]\!]_{\mathcal{M}}$
- $\mathcal{M}, w \models [B]^{\psi}\varphi$ iff for all maximal ψ-f.i.p. $\mathcal{X}^{\psi} \subseteq N(w)$, $\bigcap \mathcal{X} \cap [\![\psi]\!]_{\mathcal{M}} \subseteq [\![\varphi]\!]_{\mathcal{M}}$

Robust Belief, Reliable and Unreliable Evidence

Reliable Evidence: $N^C(w) = \{X \in N(w) \mid w \in X\}$

$\mathcal{M}, w \models \Box^C\varphi$ iff for all $v \in \bigcap N^C(w)$, $\mathcal{M}, v \models \varphi$

Unreliable Evidence: $N^U(w) = \{X \in N(w) \mid w \not\in X\}$.

$\mathcal{M}, w \models \Box^U\varphi$ iff for all $v \in \bigcup N^U(w)$, $\mathcal{M}, v \models \varphi$

Richer Languages

- Normal $+$ non-normal modalities
- First-order extensions
- Fixed-point operators/group notions (group evidence, common belief)
- Dynamic extensions (game logic, updating neighborhood models, evidence dynamics)

$$p \mid \neg\varphi \mid \varphi \wedge \psi \mid \langle\,]\varphi \mid A\varphi$$

- $\mathcal{M}, w \models i$ iff $V(i) = w$
- $\mathcal{M}, w \models \langle\,]\varphi$ iff there is a $X \in N(W)$ such that $X \subseteq [\![\varphi]\!]_{\mathcal{M}}$
- $\mathcal{M}, w \models A\varphi$ iff for all $v \in W$, $\mathcal{M}, v \models \varphi$

Non-Normal Modal Logic with a Universal Modality

$(A\text{-}K)$	$A(\varphi \rightarrow \psi) \rightarrow (A\varphi \rightarrow A\psi)$
$(A\text{-}T)$	$A\varphi \rightarrow \varphi$
$(A\text{-}4)$	$A\varphi \rightarrow AA\varphi$
$(A\text{-}B)$	$E\varphi \rightarrow AE\varphi$
$(A\text{-}Nec)$	From φ infer $A\varphi$
$(\langle\ ]\text{-}RM)$	From $\varphi \rightarrow \psi$ infer $\langle\ ]\varphi \rightarrow \langle\ ]\psi$
$(\langle\ ]\text{-}Cons)$	$\neg\langle\ ]\bot$
$(A\text{-}N)$	$A\varphi \rightarrow \langle\ ]\varphi$
$(Pullout)$	$\langle\ ](\varphi \wedge A\psi) \leftrightarrow (\langle\ ]\varphi \wedge A\psi)$

$$[\![A\alpha]\!]_{\mathfrak{M}} = \begin{cases} W & [\![\alpha]\!]_{\mathbb{M}} = W \\ \emptyset & [\![\alpha]\!]_{\mathbb{M}} \neq W \end{cases}$$

$$\langle\ ](\varphi \wedge A\psi) \leftrightarrow (\langle\ ]\varphi \wedge A\psi)$$

$$\langle\,](\varphi \wedge A\psi) \leftrightarrow (\langle\,]\varphi \wedge A\psi)$$

Case 1: $\llbracket\psi\rrbracket_{\mathcal{M}} = W$

$$\begin{array}{rcl}
\llbracket\langle\,](\varphi \wedge A\psi)\rrbracket_{\mathfrak{M}} & = & \{w \mid \exists\, X \in N(w) \text{ s.t. } X \subseteq \llbracket\varphi \wedge A\psi\rrbracket_{\mathfrak{M}}\} \\
& = & \{w \mid \exists\, X \in N(w) \text{ s.t. } X \subseteq \llbracket\varphi\rrbracket_{\mathfrak{M}} \cap \llbracket A\psi\rrbracket_{\mathfrak{M}}\} \\
& = & \{w \mid \exists\, X \in N(w) \text{ s.t. } X \subseteq \llbracket\varphi\rrbracket_{\mathfrak{M}}\} \\
& = & \llbracket\langle\,]\varphi\rrbracket_{\mathfrak{M}} \\
& = & \llbracket\langle\,]\varphi\rrbracket_{\mathfrak{M}} \cap \llbracket A\psi\rrbracket_{\mathfrak{M}} \\
& = & \llbracket\langle\,]\varphi \wedge A\psi\rrbracket_{\mathfrak{M}}
\end{array}$$

$$\langle\ ](\varphi \wedge A\psi) \leftrightarrow (\langle\ ]\varphi \wedge A\psi)$$

Case 2: $[\![\psi]\!]_{\mathcal{M}} \neq W$

Then, $[\![A\psi]\!]_{\mathfrak{M}} = \emptyset$. This implies that $[\![\varphi \wedge A\psi]\!]_{\mathfrak{M}} = [\![\langle\ ]\varphi \wedge A\psi]\!]_{\mathfrak{M}} = \emptyset$. Since, $\emptyset \not\in N(w)$ for any w, we have that $[\![\langle\ ](\varphi \wedge A\psi)]\!]_{\mathfrak{M}} = \emptyset = [\![\langle\ ]\varphi \wedge A\psi]\!]_{\mathfrak{M}}$.

Theorem. The logic EMA is sound and strongly complete with respect to neighborhood frames that are consistent, non-trivial and monotonic.

Neighborhoods with nominals

$$p \mid i \mid \neg\varphi \mid \varphi \wedge \psi \mid \Box\varphi \mid A\varphi$$

$p \in \mathsf{At}$ and $i \in \mathsf{Nom}$ (the set of nominals)

Neighborhood model with nominals $\langle W, N, V \rangle$,
$V : \mathsf{At} \cup \mathsf{Nom} \to \wp(W)$, where for all $i \in Nom$, $|V(i)| = 1$.

- $\mathfrak{M}, w \models i$ iff $V(w) = i$
- $\mathfrak{M}, w \models A\varphi$ iff for all $v \in W$, $\mathbb{M}, v \models \varphi$

$$\text{(BG)} \qquad \frac{\vdash E(i \wedge \Diamond j) \rightarrow E(j \wedge \varphi)}{\vdash E(i \wedge \Box\varphi)}$$

for $i \neq j$ and j not occurring in φ

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A class of frames F **admits** a rule provided that every model that falsifies the consequent can be extended to a model that falsifies the premises.

Theorem. A neighborhood frame is augmented iff it *admits* the rule BG.

B. ten Cate and T. Litak. *Topological Perspective on Hybrid Proof Rules*. Electronic Notes in Theoretical Computer Science, 174, pgs. 79 - 94, 2007.

First-Order Non-Normal Modal Logic

- **Barcan formula** (BF): $\forall x \Box A(x) \rightarrow \Box \forall x A(x)$
- **converse Barcan formula** (CBF): $\Box \forall x A(x) \rightarrow \forall x \Box A(x)$

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Observation. BF and CBF both valid on relational frames with constant domains

Proposition. Let $\mathcal{F}$ be a consistent constant domain neighborhood frame. The converse Barcan formula is valid on $\mathcal{F}$ iff either $\mathcal{F}$ is trivial or $\mathcal{F}$ is supplemented.

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1. $\mathcal{F}$ is trivial or
2. if D is finite, then $\mathcal{F}$ is closed under finite intersections and if D is infinite and of cardinality κ, then $\mathcal{F}$ is closed under $\leq \kappa$ intersections.

Coalition Logic: $\varphi := p \mid \neg\varphi \mid \varphi \wedge \varphi \mid [C]\varphi$

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Higher-Order Coalition Logic: $\varphi :=$
$F(x_1, \ldots, x_n) \mid Xx \mid \neg\varphi \mid \varphi \wedge \varphi \mid \forall X \varphi \mid \forall x \varphi \mid [\{x\}\varphi]\varphi \mid \langle\{x\}\varphi\rangle\varphi$

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- $F(x_1, \ldots, x_n)$ is a first-order atomic formula
- x is a first-order variable
- X is a set variable
- $\{x\}\psi$ is a group operator representing the set of all d such that $\psi[d/x]$ holds

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  Every coalition such that all of its members are users can achieve φ.

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  Every coalition such that all of its members are users can achieve φ.

- Complex relationships between coalitions and agents:
  $[\{x\}\varphi(x)]\psi \rightarrow [\{y\}\exists x(\varphi(x) \wedge \mathit{collaborates}(y,x))]\psi$

  If the coalition represented by φ can achieve ψ then so can any group that collaborates with at least one member of $\varphi(x)$.

Richer Languages

- ✓ Normal + non-normal modalities
- ✓ First-order extensions
- Dynamic extensions (game logic, updating neighborhood models, evidence dynamics)
- Fixed-point operators/group notions (group evidence, common belief)

Convention for Today

$\mathcal{M}, w \models \Box\varphi$ iff there is a $X \in N(w)$ such that $X \subseteq [\![\varphi]\!]_{\mathcal{M}}$

Background: Modeling Informational Changes

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$$[!\varphi]K\psi \quad \leftrightarrow \quad (\varphi \rightarrow K(\varphi \rightarrow [!\varphi]\psi))$$

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Accept evidence from an infallible source.

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Let $\mathcal{M} = \langle W, E, V \rangle$ be an evidence model and $\varphi \in \mathcal{L}$ a formula. The model $\mathcal{M}^{!\varphi} = \langle W^{!\varphi}, E^{!\varphi}, V^{!\varphi} \rangle$ is defined as follows: $W^{!\varphi} = [\![\varphi]\!]_{\mathcal{M}}$, for each $p \in \mathrm{At}$, $V^{!\varphi}(p) = V(p) \cap W^{!\varphi}$ and for all $w \in W$,

$$E^{!\varphi}(w) = \{X \mid \emptyset \neq X = Y \cap [\![\varphi]\!]_{\mathcal{M}} \text{ for some } Y \in E(w)\}.$$

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$[!\varphi]\psi$: “ψ is true after the public announcement of φ”

$\mathcal{M}, w \models [!\varphi]\psi$ iff $\mathcal{M}, w \models \varphi$ implies $\mathcal{M}^{!\varphi}, w \models \psi$

Public Announcements: Recursion Axioms

$$
\begin{array}{lcl}
[!\varphi]p & \leftrightarrow & (\varphi \rightarrow p) \qquad (p \in \mathsf{At}) \\
[!\varphi](\psi \wedge \chi) & \leftrightarrow & ([!\varphi]\psi \wedge [!\varphi]\chi) \\
[!\varphi]\neg\psi & \leftrightarrow & (\varphi \rightarrow \neg[!\varphi]\psi) \\
[!\varphi]\Box\psi & \leftrightarrow & (\varphi \rightarrow \Box^{\varphi}[!\varphi]\psi) \\
[!\varphi]B\psi & \leftrightarrow & (\varphi \rightarrow B^{\varphi}[!\varphi]\psi) \\
[!\varphi]\Box^{\alpha}\psi & \leftrightarrow & (\varphi \rightarrow \Box^{\varphi\wedge[!\varphi]\alpha}[!\varphi]\psi) \\
[!\varphi]B^{\alpha}\psi & \leftrightarrow & (\varphi \rightarrow B^{\varphi\wedge[!\varphi]\alpha}[!\varphi]\psi) \\
[!\varphi]A\psi & \leftrightarrow & (\varphi \rightarrow A[!\varphi]\psi)
\end{array}
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$$[!\varphi]\neg\psi \leftrightarrow (\varphi \rightarrow \neg[!\varphi]\psi)$$

$$[!\varphi]\Box\psi \leftrightarrow (\varphi \rightarrow \Box^{\varphi}[!\varphi]\psi)$$

$$[!\varphi]B\psi \leftrightarrow (\varphi \rightarrow B^{\varphi}[!\varphi]\psi)$$

$$[!\varphi]\Box^{\alpha}\psi \leftrightarrow (\varphi \rightarrow \Box^{\varphi \wedge [!\varphi]\alpha}[!\varphi]\psi)$$

$$[!\varphi]B^{\alpha}\psi \leftrightarrow (\varphi \rightarrow B^{\varphi \wedge [!\varphi]\alpha}[!\varphi]\psi)$$

$$[!\varphi]A\psi \leftrightarrow (\varphi \rightarrow A[!\varphi]\psi)$$

1. Other definition of public announcement
2. Dissecting the public announcement operation

Public Announcement

Suppose that $\mathcal{M} = \langle W, N, V \rangle$ is a monotonic neighborhood modeland $\emptyset \neq X \subseteq W$.

Intersection submodel
$N^{\Cap X}(w) = \{Y \mid \emptyset \neq Y = X \cap Z \text{ for some } Z \in N(w)\}$

Strong intersection submodel:
$N^{\cap X}(w) = \{Y \mid Y = Z \cap X \text{ for some } Z \in N(w)\}$.

Subset submodel: $N^{\subseteq X}(w) = \{Y \mid Y \subseteq X \text{ and } Y \in N(w)\}$.

- $[\varphi]^{\cap}\Box\psi \leftrightarrow (\varphi \rightarrow \Box[\varphi]^{\cap}\psi)$ is **valid** on monotonic frames.

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- Suppose that $\mathcal{M} = \langle W, N, V\rangle$ is augmented. Then, for any formula φ, $\mathcal{M}^{\cap\varphi} = \mathcal{M}^{\subseteq\varphi}$.

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- $[\varphi]^{\Cap}\Box\psi \leftrightarrow (\varphi \rightarrow \Box^{\varphi}[\varphi]^{\Cap}\psi)$ is **valid** on monotonic frames.

- $[\varphi]^{\cap}\Box\psi \leftrightarrow (\varphi \to \Box[\varphi]^{\cap}\psi)$ is **valid** on monotonic frames.

- $[\varphi]^{\subseteq}\Box\psi \leftrightarrow (\varphi \to \Box\langle\varphi\rangle^{\subseteq}\psi)$ is **valid** on monotonic frames.

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- $[\varphi]^{\Cap}\Box\psi \leftrightarrow (\varphi \to \Box^{\varphi}[\varphi]^{\Cap}\psi)$ is **valid** on monotonic frames.

- $[\varphi]^{\Cap}\Box^{\alpha}\psi \leftrightarrow (\varphi \to \Box^{\varphi\wedge[\varphi]^{\Cap}\alpha}[\varphi]^{\Cap}\psi)$ is **valid** on monotonic frames.

Dissecting the Public Announcement Operation

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Dissecting the Public Announcement Operation

On evidence models, a **public announcement** ($!\varphi$) is a complex combination of three distinct epistemic operations:

1. **Evidence addition**: accepting that φ is a piece of evidence
2. **Evidence removal**: remove evidence for $\neg\varphi$
3. **Evidence modification**: incorporate φ into each piece of evidence gathered so far

Evidence Addition

Let $\mathcal{M} = \langle W, E, V \rangle$ be an evidence model, and φ a formula in $\mathcal{L}$. The model $\mathcal{M}^{+\varphi} = \langle W^{+\varphi}, E^{+\varphi}, V^{+\varphi} \rangle$ has $W^{+\varphi} = W$, $V^{+\varphi} = V$ and for all $w \in W$,

$$E^{+\varphi}(w) = E(w) \cup \{[\![\varphi]\!]_{\mathcal{M}}\}$$

$[+\varphi]\psi$: "ψ is true after φ is accepted as an admissible piece of evidence"

$\mathcal{M}, w \models [+\varphi]\psi$ iff $\mathcal{M}, w \models E\varphi$ implies $\mathcal{M}^{+\varphi}, w \models \psi$

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Evidence Addition: Recursion Axioms

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[+\varphi]p & \leftrightarrow & (E\varphi \rightarrow p) & (p \in \mathsf{At}) \\
[+\varphi](\psi \wedge \chi) & \leftrightarrow & ([+\varphi]\psi \wedge [+\varphi]\chi) & \\
[+\varphi]\neg\psi & \leftrightarrow & (E\varphi \rightarrow \neg[+\varphi]\psi) & \\
[+\varphi]A\psi & \leftrightarrow & (E\varphi \rightarrow A[+\varphi]\psi) &
\end{array}
$$

Evidence Addition: Recursion Axioms

$$
\begin{array}{lll}
[+\varphi]\Box\psi & \leftrightarrow & (E\varphi \to (\Box[+\varphi]\psi \vee A(\varphi \to [+\varphi]\psi))) \\
[+\varphi]\Box^{\alpha}\psi & \leftrightarrow & (E\varphi \to (\Box^{[+\varphi]\alpha}[+\varphi]\psi \vee (E(\varphi \wedge [+\varphi]\alpha) \wedge \\
 & & A((\varphi \wedge [+\varphi]\alpha) \to [+\varphi]\psi))))
\end{array}
$$

Evidence Addition: Recursion Axioms

$$[+\varphi]\Box\psi \quad\leftrightarrow\quad (E\varphi \rightarrow (\Box[+\varphi]\psi \vee A(\varphi \rightarrow [+\varphi]\psi)))$$

$$[+\varphi]\Box^{\alpha}\psi \quad\leftrightarrow\quad (E\varphi \rightarrow (\Box^{[+\varphi]\alpha}[+\varphi]\psi \vee (E(\varphi \wedge [+\varphi]\alpha) \wedge A((\varphi \wedge [+\varphi]\alpha) \rightarrow [+\varphi]\psi))))$$

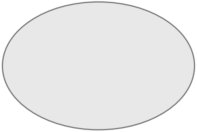
Evidence Addition: Recursion Axioms

$$[+\varphi]B\psi \quad \leftrightarrow \quad ????$$

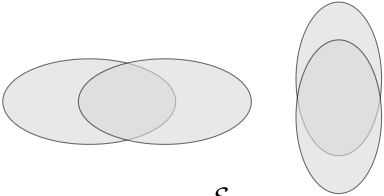
$$[+\varphi]B^{\alpha}\psi \quad \leftrightarrow \quad ????$$

Adding φ

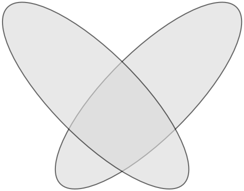
Adding φ



$\mathcal{E}_1$

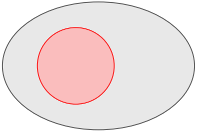


$\mathcal{E}_2$

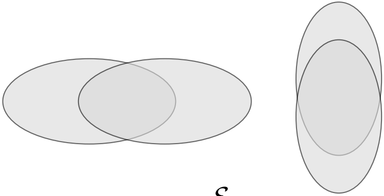


$\mathcal{E}_3$

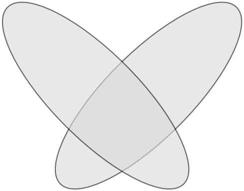
Adding φ



$\mathcal{E}_1^{+\varphi}$



$\mathcal{E}_2$



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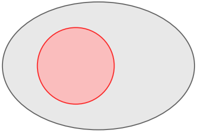
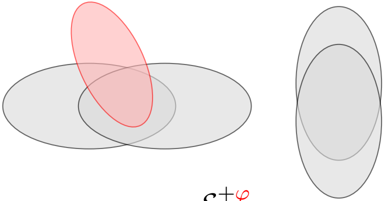
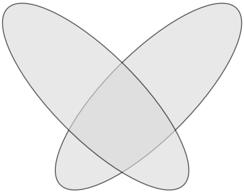
Adding φ



$\mathcal{E}_1^{+\varphi}$

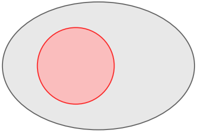


$\mathcal{E}_2^{+\varphi}$

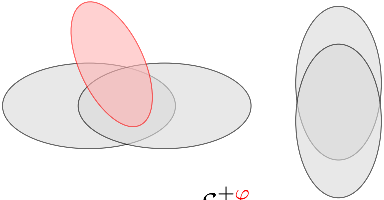


$\mathcal{E}_3$

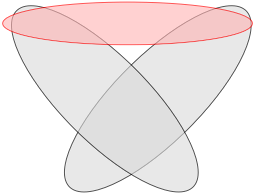
Adding φ



$\mathcal{E}_1^{+\varphi}$



$\mathcal{E}_2^{+\varphi}$



$\mathcal{E}_3^{+\varphi}$

Compatibile vs. Incompatible

Compatibile vs. Incompatible

1. $\mathcal{X}$ is maximally **φ-compatible** provided $\cap\mathcal{X} \cap [\![\varphi]\!]_{\mathcal{M}} \neq \emptyset$ and no proper extension $\mathcal{X}'$ of $\mathcal{X}$ has this property; and

Compatibile vs. Incompatible

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2. $\mathcal{X}$ is **incompatible** with φ provided there are $X_1, \ldots, X_n \in \mathcal{X}$ such that $X_1 \cap \cdots \cap X_n \subseteq [\![\neg\varphi]\!]_{\mathcal{M}}$.

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Conditional belief: $B^{+\varphi}\psi$ iff for each maximally φ-compatible $\mathcal{X} \subseteq E(w)$, $\bigcap \mathcal{X} \cap [\![\varphi]\!]_{\mathcal{M}} \subseteq [\![\psi]\!]_{\mathcal{M}}$

Conditional Beliefs (Incompatibility Version): $\mathcal{M}, w \models B^{-\varphi}\psi$ iff for all maximal f.i.p., if $\mathcal{X}$ is incompatible with φ then $\bigcap \mathcal{X} \subseteq [\![\psi]\!]_{\mathcal{M}}$.

$B^{+\neg\varphi}$ vs. $B^{-\varphi}$

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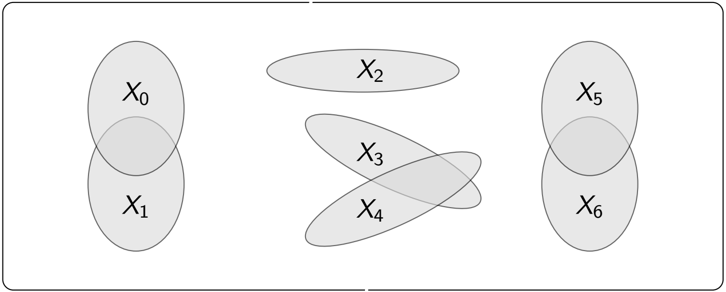
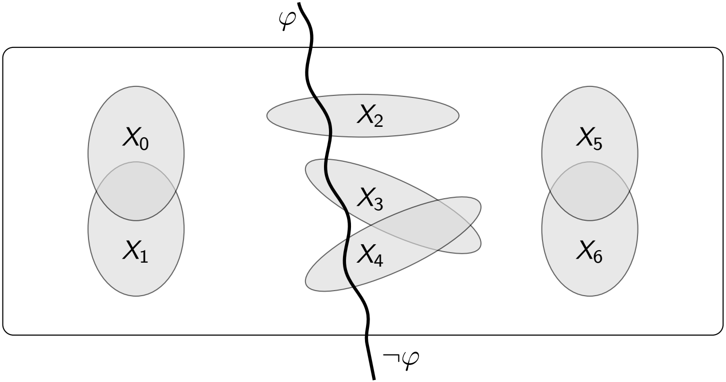
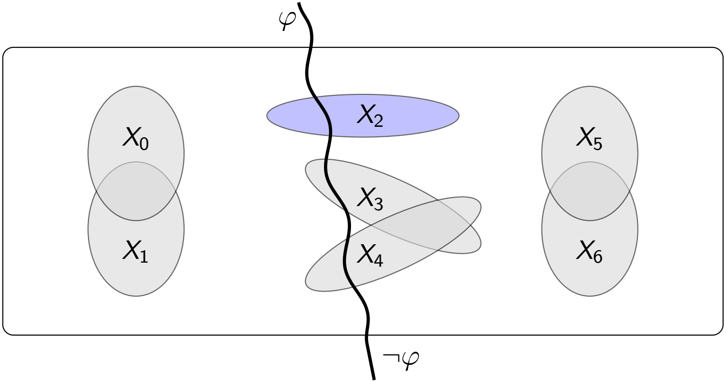


$B^{+\neg\varphi}$ vs. $B^{-\varphi}$



$B^{+\neg\varphi}$ vs. $B^{-\varphi}$



$\{X_2\}$ is (max.) compatible with $\neg\varphi$ but not maximally φ incompatible

Recursion Axiom

Fact. $[+\varphi]B\psi \leftrightarrow (E\varphi \rightarrow (B^{+\varphi}[+\varphi]\psi \wedge B^{-\varphi}[+\varphi]\psi))$ is valid.

▸ Proof Sketch

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Fact. The following is valid:

$$[+\varphi]B^{\psi,\alpha}\chi \leftrightarrow (E\varphi \to (B^{\varphi\wedge[+\varphi]\psi,[+\varphi]\alpha}[+\varphi]\chi \wedge B^{[+\varphi]\psi,\neg\varphi\wedge[+\varphi]\alpha}[+\varphi]\chi))$$

Dissecting the Public Announcement Operation

On evidence models, a **public announcement** $(!\varphi)$ is a complex combination of three distinct epistemic operations:

✓ **Evidence addition**: accepting that φ is a piece of evidence

2. **Evidence removal**: remove evidence for $\neg\varphi$

3. **Evidence modification**: incorporate φ into each piece of evidence gathered so far

Evidence Removal

Let $\mathcal{M} = \langle W, E, V \rangle$ be an evidence model, and $\varphi \in \mathcal{L}$. The model $\mathcal{M}^{-\varphi} = \langle W^{-\varphi}, E^{-\varphi}, V^{-\varphi} \rangle$ has $W^{-\varphi} = W$, $V^{-\varphi} = V$ and for all $w \in W$,

$$E^{-\varphi}(w) = E(w) - \{X \mid X \subseteq [\![\varphi]\!]_{\mathcal{M}}\}.$$

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$$E^{-\varphi}(w) = E(w) - \{X \mid X \subseteq [\![\varphi]\!]_{\mathcal{M}}\}.$$

$[-\varphi]\psi$: "after removing the evidence that φ, ψ is true"

$\mathcal{M}, w \models [-\varphi]\psi$ iff $\mathcal{M}, w \models \neg A\varphi$ implies $\mathcal{M}^{-\varphi}, w \models \psi$

Fact. Evidence removal *extends* the language.

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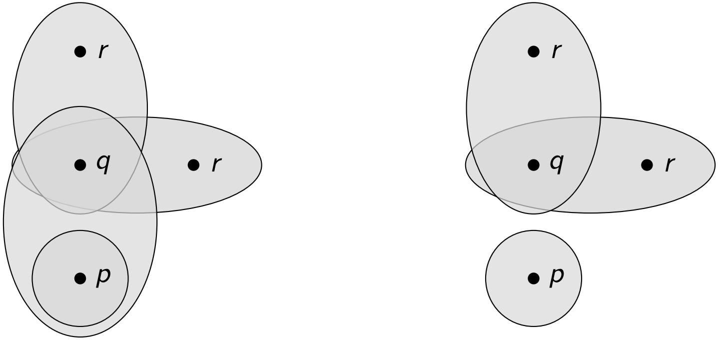

$\mathcal{E}_1$ $\mathcal{E}_2$

$[-p]\Box(p \vee q)$ is true in $\mathcal{M}_1$ but not in $\mathcal{M}_2$.

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$\Box_{\overline{\varphi}}\psi$: "$\psi$ is entailed by some admissible evidence *compatible* with each of $\overline{\varphi}$"

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$\mathcal{M}, w \models \Box_{\overline{\varphi}}\psi$ iff there is some $X \in E(w)$ compatible with $\overline{\varphi}$ where $X \subseteq [\![\psi]\!]_{\mathcal{M}}$

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Recursion axiom: $[-\varphi]\Box\psi \leftrightarrow (\neg A\varphi \rightarrow \Box_{\neg\varphi}[-\varphi]\psi)$

Evidence Removal: Recursion Axioms

Langage $\mathcal{L}'$: $p \mid \neg\varphi \mid \varphi \wedge \psi \mid B^{\alpha}_{\varphi}\psi \mid \Box^{\alpha}_{\varphi}\psi \mid A\varphi$

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$$
\begin{array}{lcl}
[-\varphi]p & \leftrightarrow & (\neg A\varphi \rightarrow p) \qquad (p \in \mathrm{At}) \\
[-\varphi](\psi \wedge \chi) & \leftrightarrow & ([-\varphi]\psi \wedge [-\varphi]\chi) \\
[-\varphi]\neg\psi & \leftrightarrow & (\neg A\varphi \rightarrow \neg[-\varphi]\psi) \\
[-\varphi]\Box^{\alpha}_{\psi}\chi & \leftrightarrow & (\neg A\varphi \rightarrow \Box^{[-\varphi]\alpha}_{[-\varphi]\overline{\psi},\neg\varphi}[-\varphi]\chi) \\
[-\varphi]B^{\alpha}_{\psi}\chi & \leftrightarrow & (\neg A\varphi \rightarrow B^{[-\varphi]\alpha}_{[-\varphi]\overline{\psi},\neg\varphi}[-\varphi]\chi) \\
[-\varphi]A\psi & \leftrightarrow & (\neg A\varphi \rightarrow A[-\varphi]\psi)
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Summary: Conditional Belief/Evidence

$\Box\psi$:	"there is evidence for ψ"
$\Box^{\varphi}\psi$:	"there is evidence compatible with φ for ψ"
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Complete logical analysis?

$$B^{\varphi}\psi \to B(\varphi \to \psi) \quad \text{and} \quad B(\varphi \to \psi) \to B^{\top,\varphi}\psi$$

Summary: Evidence Operations

Public announcement:	$[!\varphi]B\psi$	$\leftrightarrow$	$(\varphi \rightarrow B^{\varphi}[!\varphi]\psi)$
Evidence addition:	$[+\varphi]B\psi$	$\leftrightarrow$	$(E\varphi \rightarrow (B^{+\varphi}[+\varphi]\psi \wedge B^{-\varphi}[+\varphi]\psi))$
Evidence removal:	$[-\varphi]B\psi$	$\leftrightarrow$	$(\neg A\varphi \rightarrow B_{\neg\varphi}[-\varphi]\psi)$

Evidence Management

Evidence Removal: $E^{-\varphi}(w) = E(w) - \{X \mid X \subseteq [\![\varphi]\!]_{\mathcal{M}}\}$

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Evidence Modification: $E^{\oplus\varphi}(w) = \{X \cup [\![\varphi]\!]_{\mathcal{M}} \mid X \in E(w)\}$

$\mathcal{M}, w \models [\oplus\varphi]\psi$ iff $\mathcal{M}^{\oplus\varphi}, w \models \psi$

- $[\oplus\varphi]\Box\psi \leftrightarrow (\Box[\oplus\varphi]\psi \wedge A(\varphi \rightarrow [\oplus\varphi]\psi))$

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Evidence Combination: $E^{\#}(w)$ is the smallest set closed under consistent intersection and containing $E(w)$

$\mathcal{M}, w \models [\#]\varphi$ iff $\mathcal{M}^{\#}, w \models \varphi$

- Are $\neg[\#]\Box\neg\varphi \rightarrow B\varphi$ and $[\#]\Box\varphi \rightarrow B\varphi$ valid?

Evidence Combination (1)

One-round evidence combination:

$$E^{\#_1}(w) = E(w) \cup \{X \mid \text{there are } Y_1, Y_2 \in E(w) \text{ with } \emptyset \neq X = Y_1 \cap Y_2\}$$

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Is $(E(\varphi \wedge \psi) \wedge \Box\varphi \wedge \Box\psi) \rightarrow [\#_1]\Box(\varphi \wedge \psi)$ valid?

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Evidence That Operator $\mathcal{M}, w \models \boxplus\varphi$ iff $[\![\varphi]\!]_{\mathcal{M}} \in E(w)$

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Fact. $(E(\varphi \wedge \psi) \wedge \boxplus\varphi \wedge \boxplus\psi) \rightarrow [\#_1] \boxplus (\varphi \wedge \psi)$. is valid.

Evidence Combination (2)

Evidence combination Let $\mathcal{M} = \langle W, E, V \rangle$ be an evidence model. The model $\mathcal{M}^{\#} = \langle W^{\#}, E^{\#}, V^{\#} \rangle$ has $W^{\#} = W$, $V^{\#} = V$ and for all $w \in W$, $E^{\#}(w)$ is the smallest set closed under consistent intersection and containing $E(w)$.

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$[\#]\varphi$: "φ is true after the agent (consistently) combines (*all* of) her evidence"

$\mathcal{M}, w \models [\#]\varphi$ iff $\mathcal{M}^{\#}, w \models \varphi$.

Evidence Combination: Some Properties

1. $\Box[\#]\varphi \rightarrow [\#]\Box\varphi$ (combining evidence does not remove any of the original evidence)

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3. $B\varphi \rightarrow [\#]\Box\varphi$ (beliefs are explicitly supported after consistently combining evidence)

Evidence Combination: Some Properties

1. $\Box[\#]\varphi \rightarrow [\#]\Box\varphi$ (combining evidence does not remove any of the original evidence)

2. $B[\#]\varphi \leftrightarrow [\#]B\varphi$ (beliefs are immune to evidence combination)

3. $B\varphi \rightarrow [\#]\Box\varphi$ (beliefs are explicitly supported after consistently combining evidence)

4. For factual φ, $B\varphi \rightarrow \neg[\#]\Box\neg\varphi$ (if an agent believes φ then the agent cannot combine her evidence so that there is evidence for $\neg\varphi$)

Dynamically Relating Beliefs with Evidence

$B\varphi \rightarrow \Box\varphi$ vs. $B\varphi \rightarrow [\#]\Box\varphi$

Dynamically Relating Beliefs with Evidence

$B\varphi \to \Box\varphi$ vs. $B\varphi \to [\#]\Box\varphi$

$\Box\varphi \to B\varphi$ vs. $\Box\neg\varphi \to \neg B\varphi$ vs. $B\varphi \to \neg[\#]\Box\neg\varphi$

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Can we dynamically characterize beliefs in terms of evidence? Are $\neg[\#]\Box\neg\varphi \to B\varphi$ and $[\#]\Box\varphi \to B\varphi$ valid?

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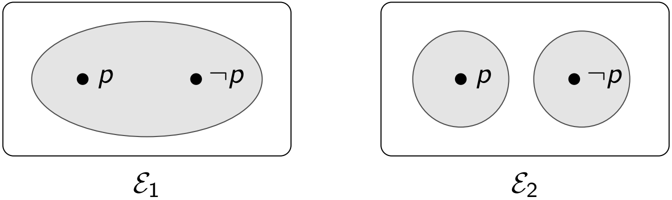
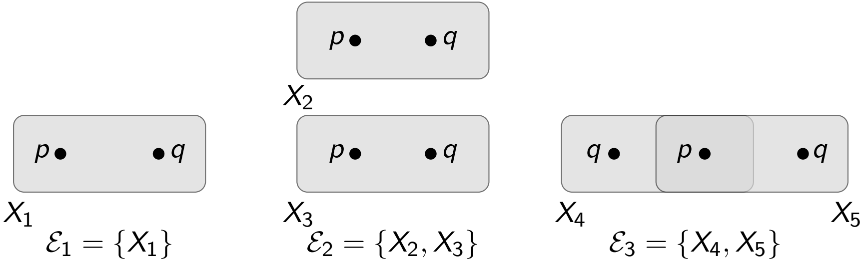
Can we dynamically characterize beliefs in terms of evidence? Are $\neg[\#]\Box\neg\varphi \to B\varphi$ and $[\#]\Box\varphi \to B\varphi$ valid? No!



Different Evidential Situations



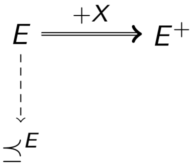
Tracking...

E

$\downarrow$

$\preceq^E$

Tracking...



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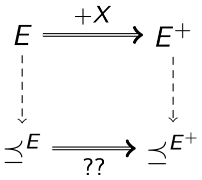
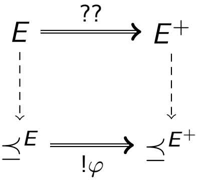


$$\preceq^{E^+} = \preceq^E - \{(w, v) \mid v \in X \text{ and } w \not\in X\}.$$

Tracking...



Richer Languages

- ✓ Normal + non-normal modalities
- ✓ First-order extensions
- Dynamics
  - ✓ Dynamics on neighborhoods (updating neighborhood models, evidence dynamics)
  - Dynamics with neighborhoods (Game logic)
- Fixed-point operators/group notions (group evidence, common belief)

Background: Propositional Dynamic Logic

Semantics: $\mathcal{M} = \langle W, \{R_a \mid a \in \mathsf{P}\}, V\rangle$ where for each $a \in \mathsf{P}$, $R_a \subseteq W \times W$ and $V : \mathsf{At} \to \wp(W)$

- $R_{\alpha\cup\beta} := R_\alpha \cup R_\beta$
- $R_{\alpha;\beta} := R_\alpha \circ R_\beta$
- $R_{\alpha^*} := \cup_{n\geq 0} R_\alpha^n$
- $R_{\varphi?} = \{(w, w) \mid \mathcal{M}, w \models \varphi\}$

$$\mathcal{M}, w \models [\alpha]\varphi \text{ iff for each } v\text{, if } wR_\alpha v \text{ then } \mathcal{M}, v \models \varphi$$

Background: Propositional Dynamic Logic

1. Axioms of propositional logic
2. $[\alpha](\varphi \rightarrow \psi) \rightarrow ([\alpha]\varphi \rightarrow [\alpha]\psi)$
3. $[\alpha \cup \beta]\varphi \leftrightarrow [\alpha]\varphi \wedge [\beta]\varphi$
4. $[\alpha; \beta]\varphi \leftrightarrow [\alpha][\beta]\varphi$
5. $[\psi?]\varphi \leftrightarrow (\psi \rightarrow \varphi)$
6. $\varphi \wedge [\alpha][\alpha^*]\varphi \leftrightarrow [\alpha^*]\varphi$
7. $\varphi \wedge [\alpha^*](\varphi \rightarrow [\alpha]\varphi) \rightarrow [\alpha^*]\varphi$
8. Modus Ponens and Necessitation (for each program α)

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5. $[\psi?]\varphi \leftrightarrow (\psi \to \varphi)$
6. $\varphi \wedge [\alpha][\alpha^*]\varphi \leftrightarrow [\alpha^*]\varphi$ (Fixed-Point Axiom)
7. $\varphi \wedge [\alpha^*](\varphi \to [\alpha]\varphi) \to [\alpha^*]\varphi$ (Induction Axiom)
8. Modus Ponens and Necessitation (for each program α)

Background: Propositional Dynamic Logic

Theorem PDL is sound and weakly complete with respect to the Segerberg Axioms.

Theorem The satisfiability problem for **PDL** is decidable ($\mathrm{EXPTIME}$-Complete).

D. Kozen and R. Parikh. *A Completeness proof for Propositional Dynamic Logic*. .

D. Harel, D. Kozen and Tiuryn. *Dynamic Logic*. 2001.

Concurrent Programs

D. Peleg. *Concurrent Dynamic Logic*. JACM (1987).

Concurrent Programs

$\alpha \cap \beta$ is intended to mean "execute α and β in parallel".

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In PDL: $R_\alpha \subseteq W \times W$, where $wR_\alpha v$ means executing α in state w leads to state v.

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With Concurrent Programs: $R_\alpha \subseteq W \times \wp(W)$, where $wR_\alpha V$ means executing α in parallel from state w to reach all states in V.

$w \models \langle\alpha\rangle\varphi$ iff $\exists U$ such that $(w, U) \in R_\alpha$ and $\forall v \in U$, $v \models \varphi$.

$R_{\alpha\cap\beta} := \{(w, V) \mid \exists U, U',\ (w, U) \in R_\alpha,\ (w, U') \in R_\beta,\ V = U \cup U'\}$

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From **PDL** to Game Logic

R. Parikh. *The Logic of Games and its Applications.*. Annals of Discrete Mathematics. (1985) .

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Main Idea:
In **PDL**: $w \models \langle\pi\rangle\varphi$: there is a run of the program π starting in state w that ends in a state where φ is true.

The programs in **PDL** can be thought of as *single player games*.

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The programs in **PDL** can be thought of as *single player games*.

Game Logic generalized **PDL** by considering two players:

In **GL**: $w \models \langle\gamma\rangle\varphi$: Angel has a **strategy** in the game γ to ensure that the game ends in a state where φ is true.

From **PDL** to Game Logic

Consequences of two players:

From **PDL** to Game Logic

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$\langle\gamma\rangle\varphi$: Angel has a strategy in γ to ensure φ is true

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Thus, $[\gamma]\varphi \leftrightarrow \neg\langle\gamma\rangle\neg\varphi$ is a valid principle

However, $[\gamma]\varphi \wedge [\gamma]\psi \rightarrow [\gamma](\varphi \wedge \psi)$ is **not** a valid principle

From **PDL** to Game Logic

Reinterpret operations and invent new ones:

- $?\varphi$: Check whether φ currently holds
- $\gamma_1;\gamma_2$: First play γ_1 then γ_2
- $\gamma_1 \cup \gamma_2$: Angel choose between γ_1 and γ_2
- γ^*: Angel can choose how often to play γ (possibly not at all); each time she has played γ, she can decide whether to play it again or not.
- γ^d: Switch roles, then play γ
- $\gamma_1 \cap \gamma_2 := (\gamma_1^d \cup \gamma_2^d)^d$: Demon chooses between γ_1 and γ_2
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Syntax

Let Γ_0 be a set of atomic games and At a set of atomic propositions. Then formulas of Game Logic are defined inductively as follows:

$$\gamma \;:=\; g \mid \varphi? \mid \gamma;\gamma \mid \gamma \cup \gamma \mid \gamma^* \mid \gamma^d$$
$$\varphi \;:=\; \bot \mid p \mid \neg\varphi \mid \varphi \vee \varphi \mid \langle\gamma\rangle\varphi \mid [\gamma]\varphi$$

where $p \in \mathrm{At}, g \in \Gamma_0$.

Game Logic

A neighborhood game model is a tuple
$\mathcal{M} = \langle W, \{E_g \mid g \in \Gamma_0\}, V\rangle$ where

W is a nonempty set of states

For each $g \in \Gamma_0$, $E_g : W \to \wp(\wp(W))$ is a monotonic neighborhood function.

$X \in E_g(w)$ means in state s, Angel has a strategy to force the game to end in *some* state in X (we may write wE_gX)

$V : \mathsf{At} \to \wp(W)$ is a valuation function.

Game Logic

Propositional letters and boolean connectives are as usual.

$\mathcal{M}, w \models \langle\gamma\rangle\varphi$ iff $[\![\varphi]\!]_{\mathcal{M}} \in E_\gamma(w)$

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$\mathcal{M}, w \models \langle\gamma\rangle\varphi$ iff $[\![\varphi]\!]_{\mathcal{M}} \in E_\gamma(w)$

Suppose $E_\gamma(Y) := \{s \mid Y \in E_g(s)\}$

- $E_{\gamma_1;\gamma_2}(Y) \ := \ E_{\gamma_1}(E_{\gamma_2}(Y))$
- $E_{\gamma_1\cup\gamma_2}(Y) \ := \ E_{\gamma_1}(Y) \cup E_{\gamma_2}(Y)$
- $E_{\varphi?}(Y) \ := \ (\varphi)^{\mathcal{M}} \cap Y$
- $E_{\gamma^d}(Y) \ := \ \overline{E_\gamma(\overline{Y})}$
- $E_{\gamma^*}(Y) \ := \ \mu X.Y \cup E_\gamma(X)$

Game Logic: Axioms

1. All propositional tautologies
2. $\langle \alpha ; \beta \rangle \varphi \leftrightarrow \langle \alpha \rangle \langle \beta \rangle \varphi$ Composition
3. $\langle \alpha \cup \beta \rangle \varphi \leftrightarrow \langle \alpha \rangle \varphi \vee \langle \beta \rangle \varphi$ Union
4. $\langle \psi ? \rangle \varphi \leftrightarrow (\psi \wedge \varphi)$ Test
5. $\langle \alpha^d \rangle \varphi \leftrightarrow \neg \langle \alpha \rangle \neg \varphi$ Dual
6. $(\varphi \vee \langle \alpha \rangle \langle \alpha^* \rangle \varphi) \rightarrow \langle \alpha^* \rangle \varphi$ Mix

and the rules,

$$\frac{\varphi \qquad \varphi \rightarrow \psi}{\psi} \qquad \frac{\varphi \rightarrow \psi}{\langle \alpha \rangle \varphi \rightarrow \langle \alpha \rangle \psi} \qquad \frac{(\varphi \vee \langle \alpha \rangle \psi) \rightarrow \psi}{\langle \alpha^* \rangle \varphi \rightarrow \psi}$$

Game Logic

- Game Logic is more expressive than **PDL**

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$$\langle (g^d)^* \rangle \bot$$

Game Logic

Theorem Dual-free game logic is sound and complete with respect to the class of all game models.

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Theorem Iteration-free game logic is sound and complete with respect to the class of all game models.

Open Question Is (full) game logic complete with respect to the class of all game models?

R. Parikh. *The Logic of Games and its Applications.*. Annals of Discrete Mathematics, 1985.

M. Pauly. *Logic for Social Software*. Ph.D. Thesis, University of Amsterdam (2001).

Game Logic

Theorem Given a game logic formula φ and a finite game model $\mathcal{M}$, model checking can be done in time $O(|\mathcal{M}|^{ad(\varphi)+1} \times |\varphi|)$

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Game Logic

Theorem The satisfiability problem for game logic is in EXPTIME.

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Game Logic

Theorem Game logic can be translated into the modal μ-calculus

Theorem No finite level of the modal μ-calculus hierarchy captures the expressive power of game logic.

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M. Pauly. *Logic for Social Software*. Ph.D. Thesis, University of Amsterdam (2001).

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Game Algebra

Definition Two games γ_1 and γ_2 are equivalent provided $E_{\gamma_1} = E_{\gamma_2}$ in all models

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Definition Two games γ_1 and γ_2 are equivalent provided $E_{\gamma_1} = E_{\gamma_2}$ in all models (iff $\langle\gamma_1\rangle p \leftrightarrow \langle\gamma_2\rangle p$ is valid for a p which occurs neither in γ_1 nor in γ_2.)

▸ Skip

Game Algebra

Game Boards: Given a set of states or positions B, for each game g and each player i there is an associated relation $E^i_g \subseteq B \times 2^B$:

$pE^i_g T$ holds if in position p, i can force that the outcome of g will be a position in T.

- (monotonicity) if $pE^i_g T$ and $T \subseteq U$ then $pE^i_g U$
- (consistency) if $pE^i_g T$ then not $pE^{1-i}_g(B - T)$

Given a game board (a set B with relations E^i_g for each game and player), we say that two games g, h ($g \approx h$) are equivalent if $E^i_g = E^i_h$ for each i.

Game Algebra

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1. Standard Laws of Boolean Algebras

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4. $-x; -y \approx -(x; y)$

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4. $-x; -y \approx -(x; y)$
5. $y \preceq z \Rightarrow x; y \preceq x; z$

Theorem Sound and complete axiomatizations of (iteration free) game algebra

Y. Venema. *Representing Game Algebras*. Studia Logica **75** (2003).

V. Goranko. *The Basic Algebra of Game Equivalences*. Studia Logica **75** (2003).

Concurrent Game Logic

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Need both the disjunctive and conjunctive interpretation of the neighborhoods.

Main Idea: $R_\gamma \subseteq W \times \wp(\wp(\wp(W)))$

J. van Benthem, S. Ghosh and F. Liu. *Modelling Simultaneous Games in Dynamic Logic*. Synthese, 165(2), pgs. 247-268, 2008.

Richer Languages

- ✓ Normal + non-normal modalities
- ✓ First-order extensions
- Dynamics
  - ✓ Dynamics on neighborhoods (updating neighborhood models, evidence dynamics)
  - ✓ Dynamics with neighborhoods (Game logic)
- Fixed-point operators/group notions (group evidence, common belief)

Multi-Agent Evidence Logic

Social notions: Let $\mathcal{M} = \langle W, \mathcal{E}_i, \mathcal{E}_j, V\rangle$ be a multiagent evidence model. What evidence does the group i, j have?

- $\mathcal{M}, w \models \Box^{\{i,j\}}\varphi$ iff there is a $X \in \mathcal{E}_i \cup \mathcal{E}_j$ such that $X \subseteq [\![\varphi]\!]_{\mathcal{M}}$
- $\mathcal{M}, w \models \Box_{\{i,j\}}\varphi$ iff there is a $X \in \mathcal{E}_i \cap \mathcal{E}_j$ such that $X \subseteq [\![\varphi]\!]_{\mathcal{M}}$
- $\mathcal{M}, w \models [i \sqcap j]\varphi$ iff there exists $X \in \mathcal{E}_i \sqcap \mathcal{E}_j$ with $X \subseteq [\![\varphi]\!]_{\mathcal{M}}$
  $\mathcal{E}_i \sqcap \mathcal{E}_j = \{ Y \mid \emptyset \neq Y = X \cap X'$ with $X \in \mathcal{E}_i$ and $X' \in \mathcal{E}_j \}$

Thank You!

Proof Sketch

$\mathcal{M}, w \models [+\varphi]B\psi$ iff
for each maximally f.i.p. $\mathcal{X} \subseteq E^{+\varphi}(w)$, $\cap\mathcal{X} \subseteq [\![\psi]\!]_{\mathcal{M}^{+\varphi}}$

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1. $[\![\varphi]\!]_{\mathcal{M}} \in \mathcal{X}$.

Proof Sketch

$\mathcal{M}, w \models [+\varphi]B\psi$ iff
for each maximally f.i.p. $\mathcal{X} \subseteq E^{+\varphi}(w)$, $\cap\mathcal{X} \subseteq [\![\psi]\!]_{\mathcal{M}^{+\varphi}}$

1. $[\![\varphi]\!]_{\mathcal{M}} \in \mathcal{X}$. Then $\mathcal{X} - \{[\![\varphi]\!]_{\mathcal{M}}\}$ is a maximal f.i.p. in $\mathcal{M}$ that *is compatible with* φ.

$$\bigcap \mathcal{X}^{\varphi} = \bigcap \mathcal{X} \subseteq [\![\psi]\!]_{\mathcal{M}^{+\varphi}} = [\![[+\varphi]\psi]\!]_{\mathcal{M}}$$

Proof Sketch

$\mathcal{M}, w \models [+\varphi]B\psi$ iff
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Proof Sketch

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2. $\mathcal{X}$ is incompatible with φ: there exists $X_1 \ldots, X_n \in \mathcal{X}$ such that $X_1 \cap \cdots X_n \subseteq [\![\neg\varphi]\!]_{\mathcal{M}}$. Then $\mathcal{X}$ is a maximal f.i.p. in $\mathcal{M}$ with

$$\bigcap \mathcal{X} \subseteq [\![\psi]\!]_{\mathcal{M}^{+\varphi}} = [\![[+\varphi]\psi]\!]_{\mathcal{M}}$$

▸ Back

Proof Sketch

$\mathcal{M}, w \models [+\varphi]B\psi$ iff $\mathcal{M}, w \models B^{+\varphi}[+\varphi]\psi \wedge B^{-\varphi}[+\varphi]\psi$
for each maximally f.i.p. $\mathcal{X} \subseteq E^{+\varphi}(w)$, $\cap\mathcal{X} \subseteq [\![\psi]\!]_{\mathcal{M}^{+\varphi}}$

1. $[\![\varphi]\!]_{\mathcal{M}} \in \mathcal{X}$. Then $\mathcal{X} - \{[\![\varphi]\!]_{\mathcal{M}}\}$ is a maximal f.i.p. in $\mathcal{M}$ that *is compatible with* φ.

$$\bigcap \mathcal{X}^{\varphi} = \bigcap \mathcal{X} \subseteq [\![\psi]\!]_{\mathcal{M}^{+\varphi}} = [\![[+\varphi]\psi]\!]_{\mathcal{M}}$$

   $\mathcal{M}, w \models B^{+\varphi}[+\varphi]\psi$

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$$\bigcap \mathcal{X} \subseteq [\![\psi]\!]_{\mathcal{M}^{+\varphi}} = [\![[+\varphi]\psi]\!]_{\mathcal{M}}$$

   $\mathcal{M}, w \models B^{-\varphi}[+\varphi]\psi$

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Lecture 8: June 14th, 10h00-13h00

Lecture 9 Discussion of the course, problems/future research/: June 15th, 10h00-11h30