Neighborhood Semantics for Modal Logic
Lecture 6

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Text

Neighborhood semantics for modal logic (Draft)

Ch 1: Introduction and Motivation

Ch 2: Core Theory: Expressivity, Completeness, Decidability, Complexity, Correspondence Theory

Ch 3: Richer Languages: Fixed-point operators, First-order extensions, Dynamic operators

Schedule

Lecture 1: June 1st, 14h00-16h30

Lecture 2: June 2nd 12h30-14h30

Lecture 3: June 7th, 14h00-16h30

Lecture 4: June 8th, 11h00-13h00

Lecture 5: June 8th, 14h00-16h30

Lecture 7: June 9th, 12h30-14h30

Lecture 8: June 13th,12h30-15h00

Lecture 9: June 14th, 10h00-13h00

Lecture 10 Presentations (solutions to problems etc.): June 15th, 10h00-13h00

Core Theory

- ✓ Expressivity: Bisimulation, Behavioral Equivalence
- ✓ Completeness, Decidability, Complexity
- ✓ Alternative Semantics for Non-Normal Modal Logics
- ✓ Incompleteness
- ✓ From Neighborhoods to Orders
- ✓ Simulating Non-Normal Modal Logics

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- ✓ Simulating Non-Normal Modal Logics
- Proof Theory (Tableaux, Sequents, etc.)
- Model Theoretic Results: Sahlqvist Theorem, van Benthem Characterization Theorem, Iterpolation

Richer Languages

$$p \mid \neg\varphi \mid \varphi \wedge \psi \mid \Box\varphi \mid \langle\ ]\varphi$$

- $\mathcal{M}, w \models \Box\varphi$ iff $[\![\varphi]\!]_{\mathcal{M}} \in N(W)$
- $\mathcal{M}, w \models \langle\ ]\varphi$ iff there is a $X \in N(W)$ such that $X \subseteq [\![\varphi]\!]_{\mathcal{M}}$

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- $\mathcal{M}, w \models \langle\,]^{\psi}\varphi$ there is a $X \in N(W)$ such that $X \cap [\![\psi]\!]_{\mathcal{M}} \neq \emptyset$ and $X \cap [\![\psi]\!]_{\mathcal{M}} \subseteq [\![\varphi]\!]_{\mathcal{M}}$.

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- $\mathcal{M}, w \models \langle\,]^{\psi}\varphi$ there is a $X \in N(W)$ such that $X \cap [\![\psi]\!]_{\mathcal{M}} \neq \emptyset$ and $X \cap [\![\psi]\!]_{\mathcal{M}} \subseteq [\![\varphi]\!]_{\mathcal{M}}$.
- $\mathcal{M}, w \models [B]\varphi$ iff for all max-f.i.p. $\mathcal{X} \subseteq N(w)$, $\bigcap \mathcal{X} \subseteq [\![\varphi]\!]_{\mathcal{M}}$
- $\mathcal{M}, w \models [B]^{\psi}\varphi$ iff for all maximal ψ-f.i.p. $\mathcal{X}^{\psi} \subseteq N(w)$, $\bigcap \mathcal{X} \cap [\![\psi]\!]_{\mathcal{M}} \subseteq [\![\varphi]\!]_{\mathcal{M}}$

Robust Belief, Reliable and Unreliable Evidence

Robust Belief: $\mathcal{M}, w \models B^r\varphi$ iff for each $X \subseteq W$ with $w \in X$, we have $Min_{\preceq}(X) \subseteq [\![\varphi]\!]_{\mathcal{M}}$

Reliable Evidence: $N^C(w) = \{X \in N(w) \mid w \in X\}$

$\mathcal{M}, w \models \Box^C\varphi$ iff for all $v \in \bigcap N^C(w)$, $\mathcal{M}, v \models \varphi$

Unreliable Evidence: $N^U(w) = \{X \in N(w) \mid w \notin X\}$.

$\mathcal{M}, w \models \Box^U\varphi$ iff for all $v \in \bigcup N^U(w)$, $\mathcal{M}, v \models \varphi$

Richer Languages

- Normal + non-normal modalities
- First-order extensions
- Fixed-point operators/group notions (group evidence, common belief)
- Dynamic extensions (game logic, updating neighborhood models, evidence dynamics)

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- $\mathcal{M}, w \models A\varphi$ iff for all $v \in W$, $\mathcal{M}, v \models \varphi$

Non-Normal Modal Logic with a Universal Modality

$(A\text{-}K)$	$A(\varphi \rightarrow \psi) \rightarrow (A\varphi \rightarrow A\psi)$
$(A\text{-}T)$	$A\varphi \rightarrow \varphi$
$(A\text{-}4)$	$A\varphi \rightarrow AA\varphi$
$(A\text{-}B)$	$E\varphi \rightarrow AE\varphi$
$(A\text{-}Nec)$	From φ infer $A\varphi$

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$(A\text{-}Nec)$	From φ infer $A\varphi$
$(\langle\,]\text{-}RM)$	From $\varphi \rightarrow \psi$ infer $\langle\,]\varphi \rightarrow \langle\,]\psi$
$(\langle\,]\text{-}Cons)$	$\neg\langle\,]\bot$
$(A\text{-}N)$	$A\varphi \rightarrow \langle\,]\varphi$
$(Pullout)$	$\langle\,](\varphi \wedge A\psi) \leftrightarrow (\langle\,]\varphi \wedge A\psi)$

$$[\![A\alpha]\!]_{\mathfrak{M}} = \begin{cases} W & [\![\alpha]\!]_{\mathbb{M}} = W \\ \emptyset & [\![\alpha]\!]_{\mathbb{M}} \neq W \end{cases}$$

$$\langle\ ](\varphi \wedge A\psi) \leftrightarrow (\langle\ ]\varphi \wedge A\psi)$$

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Case 1: $\llbracket \psi \rrbracket_{\mathcal{M}} = W$

$$\begin{array}{rcl}
\llbracket \langle\ ](\varphi \wedge A\psi) \rrbracket_{\mathfrak{M}} & = & \{w \mid \exists\, X \in N(w) \text{ s.t. } X \subseteq \llbracket \varphi \wedge A\psi \rrbracket_{\mathfrak{M}}\} \\
& = & \{w \mid \exists\, X \in N(w) \text{ s.t. } X \subseteq \llbracket \varphi \rrbracket_{\mathfrak{M}} \cap \llbracket A\psi \rrbracket_{\mathfrak{M}}\} \\
& = & \{w \mid \exists\, X \in N(w) \text{ s.t. } X \subseteq \llbracket \varphi \rrbracket_{\mathfrak{M}}\} \\
& = & \llbracket \langle\ ]\varphi \rrbracket_{\mathfrak{M}} \\
& = & \llbracket \langle\ ]\varphi \rrbracket_{\mathfrak{M}} \cap \llbracket A\psi \rrbracket_{\mathfrak{M}} \\
& = & \llbracket \langle\ ]\varphi \wedge A\psi \rrbracket_{\mathfrak{M}}
\end{array}$$

$$\langle\ ](\varphi \wedge A\psi) \leftrightarrow (\langle\ ]\varphi \wedge A\psi)$$

Case 2: $[\![\psi]\!]_{\mathcal{M}} \neq W$

Then, $[\![A\psi]\!]_{\mathfrak{M}} = \emptyset$. This implies that $[\![\varphi \wedge A\psi]\!]_{\mathfrak{M}} = [\![\langle\ ]\varphi \wedge A\psi]\!]_{\mathfrak{M}} = \emptyset$. Since, $\emptyset \not\in N(w)$ for any w, we have that $[\![\langle\ ](\varphi \wedge A\psi)]\!]_{\mathfrak{M}} = \emptyset = [\![\langle\ ]\varphi \wedge A\psi]\!]_{\mathfrak{M}}$.

Theorem. The logic EMA is sound and strongly complete with respect to neighborhood frames that are consistent, non-trivial and monotonic.

Neighborhoods with nominals

$$p \mid i \mid \neg\varphi \mid \varphi \wedge \psi \mid \Box\varphi \mid A\varphi$$

$p \in \mathsf{At}$ and $i \in \mathsf{Nom}$ (the set of nominals)

Neighborhood model with nominals $\langle W, N, V \rangle$,
$V : \mathsf{At} \cup \mathsf{Nom} \rightarrow \wp(W)$, where for all $i \in Nom$, $|V(i)| = 1$.

- $\mathfrak{M}, w \models i$ iff $V(w) = i$
- $\mathfrak{M}, w \models A\varphi$ iff for all $v \in W$, $\mathbb{M}, v \models \varphi$

$$\text{(BG)} \qquad \frac{\vdash E(i \wedge \Diamond j) \rightarrow E(j \wedge \varphi)}{\vdash E(i \wedge \Box \varphi)}$$

for $i \neq j$ and j not occurring in φ

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A class of frames F **admits** a rule provided that every model that falsifies the consequent can be extended to a model that falsifies the premises.

Theorem. A neighborhood frame is augmented iff it *admits** the rule BG.

B. ten Cate and T. Litak. *Topological Perspective on Hybrid Proof Rules*. Electronic Notes in Theoretical Computer Science, 174, pgs. 79 - 94, 2007.

Characterizing Augmented Frames

Theorem. A neighborhood frame is augmented iff it *admits** the rule BG.

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* A class of frames admits a rule provided every falsifying model of the consequent can be *extended* to a falsifying model of the premises.

Richer Languages

- ✓ Normal + non-normal modalities
- First-order extensions
- Fixed-point operators/group notions (group evidence, common belief)
- Dynamic extensions (game logic, updating neighborhood models, evidence dynamics)

Neighborhood Models for First-Order Modal Logic

H. Arlo Costa and E. Pacuit. *First-Order Classical Modal Logic*. Studia Logica, **84**, pgs. 171 - 210 (2006).

Higher-Order Coalition Logic (time permitting)

G. Boella, D. Gabbay, V. Genovese, L. van der Torre. *Higher-Order Coalition Logic*. 2010.

First-Order Modal Language: $\mathcal{L}_1$

Extend the propositional modal language $\mathcal{L}$ with the usual first-order machinery (constants, terms, predicate symbols, quantifiers).

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$$A := P(t_1, \ldots, t_n) \mid \neg A \mid A \wedge A \mid \Box A \mid \forall x A$$

(note that equality is not in the language!)

State-of-the-art

T. Braüner and S. Ghilardi. *First-order Modal Logic*. Handbook of Modal Logic, pgs. 549 - 620 (2007).

D.Gabbay, V. Shehtman and D. Skvortsov. *Quantification in Nonclassical Logic*. Draft available (2008).
http://lpcs.math.msu.su/~shehtman/QNCLfinal.pdf

M. Fitting and R. Mendelsohn. *First-Order Modal Logic*. Kluwer Academic Publishers (1998).

First-order Modal Logic

A **constant domain Kripke frame** is a tuple $\langle W, R, D\rangle$ where W and D are sets, and $R \subseteq W \times W$.

A **constant domain Kripke model** adds a valuation function I, where for each n-ary relation symbol P and $w \in W$, $I(P, w) \subseteq D^n$.

A **substitution** is any function $\sigma : \mathcal{V} \to D$ ($\mathcal{V}$ the set of variables).

A substitution σ' is said to be an x-**variant** of σ if $\sigma(y) = \sigma'(y)$ for all variable y except possibly x, this will be denoted by $\sigma \sim_x \sigma'$.

First-order Modal Logic

A **constant domain Kripke frame** is a tuple $\langle W, R, D\rangle$ where W and D are sets, and $R \subseteq W \times W$.

A **constant domain Kripke model** adds a valuation function V, where for each n-ary relation symbol P and $w \in W$, $I(P, w) \subseteq D^n$.

Suppose that σ is a substitution.

1. $\mathcal{M}, w \models_\sigma P(x_1, \ldots, x_n)$ iff $\langle \sigma(x_1), \ldots, \sigma(x_n)\rangle \in I(P, w)$
2. $\mathcal{M}, w \models_\sigma \Box A$ iff $R(w) \subseteq [\![\varphi]\!]_{\mathcal{M},\sigma}$
3. $\mathcal{M}, w \models_\sigma \forall x A$ iff for each x-variant σ', $\mathcal{M}, w \models_{\sigma'} A$

First-order Modal Logic

A **constant domain Neighborhood frame** is a tuple $\langle W, N, D\rangle$ where W and D are sets, and $N : W \to \wp(\wp(W))$.

A **constant domain Neighborhood model** adds a valuation function V, where for each n-ary relation symbol P and $w \in W$, $I(P, w) \subseteq D^n$.

Suppose that σ is a substitution.

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2. $\mathcal{M}, w \models_\sigma \Box A$ iff $[\![\varphi]\!]_{\mathcal{M},\sigma} \in N(w)$
3. $\mathcal{M}, w \models_\sigma \forall x A$ iff for each x-variant σ', $\mathcal{M}, w \models_{\sigma'} A$

Example

Suppose that F is a unary predicate symbol, $\mathcal{V} = \{x, y\}$, and $\langle W, N, D, I\rangle$ is a first order constant domain neighborhood model where

- $W = \{w, v, u\}$;
- $N(w) = \{\{w, v\}, \{u\}\}$, $N(v) = \{\{v\}\}$, $N(u) = \{\{w, v\}, \{v\}\}$;
- $D = \{a, b\}$; and
- $I(F, w) = \{a\}$, $I(F, v) = \{a, b\}$, and $I(F, u) = \emptyset$.

Example

There are four possible substitutions:

- $\sigma_1 : \mathcal{V} \to D$ where $\sigma_1(x) = a$, $\sigma_1(y) = b$;
- $\sigma_2 : \mathcal{V} \to D$ where $\sigma_2(x) = b$, $\sigma_2(y) = a$;
- $\sigma_3 : \mathcal{V} \to D$ where $\sigma_3(x) = \sigma_3(y) = a$; and
- $\sigma_4 : \mathcal{V} \to D$ where $\sigma_4(x) = \sigma_4(y) = b$

- $[\![F(x)]\!]_{\mathcal{M},\sigma_1} = \{w, v\}$;
- $[\![F(x)]\!]_{\mathcal{M},\sigma_2} = \{v\}$;
- $[\![F(x)]\!]_{\mathcal{M},\sigma_3} = \{w, v\}$; and
- $[\![F(x)]\!]_{\mathcal{M},\sigma_4} = \{v\}$.

Example

In general, every formula $\varphi \in \mathcal{L}_1$ is associated with a function

$$[\![\varphi]\!] : D^{\mathcal{V}} \to \wp(W)$$

Example

- $[\![\Box F(x)]\!]_{\mathcal{M},\sigma_1} = [\![\Box F(x)]\!]_{\mathcal{M},\sigma_3} = \{w, u\}$
  $[\![\Box F(x)]\!]_{\mathcal{M},\sigma_2} = [\![\Box F(x)]\!]_{\mathcal{M},\sigma_4} = \{v, u\}$;
- $[\![\Box \forall x F(x)]\!]_{\mathcal{M},\sigma_1} = \{v\}$; and
- $[\![\forall x \Box F(x)]\!]_{\mathcal{M},\sigma_1} = \{v, u\}$.

Barcan Schemas

- **Barcan formula** (BF): $\forall x \Box A(x) \rightarrow \Box \forall x A(x)$
- **converse Barcan formula** (CBF): $\Box \forall x A(x) \rightarrow \forall x \Box A(x)$

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- **converse Barcan formula** (CBF): $\Box \forall x A(x) \rightarrow \forall x \Box A(x)$

Observation 1: CBF is provable in $\mathbf{FOL} + \mathbf{EM}$

Observation 2: BF and CBF both valid on relational frames with constant domains

Observation 3: BF is valid in a *varying* domain relational frame iff the frame is anti-monotonic; CBF is valid in a *varying* domain relational frame iff the frame is monotonic.

See (Fitting and Mendelsohn, 1998) for an extended discussion

Constant Domains without the Barcan Formula

The system **EMN** and seems to play a central role in characterizing monadic operators of high probability (See Kyburg and Teng 2002, Arló-Costa 2004).

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Of course, *BF* should fail in this case, given that it instantiates cases of what is usually known as the '**lottery paradox**':

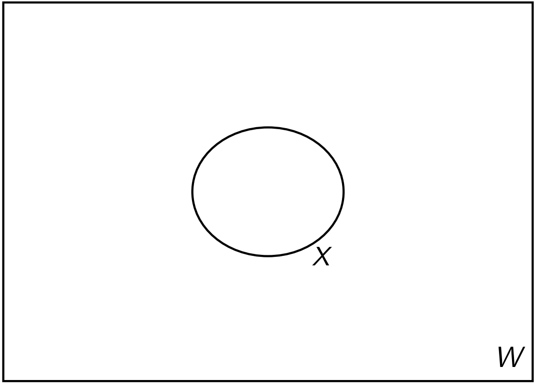
For each individual x, it is *highly probably* that x will loose the lottery; however it is not necessarily highly probably that each individual will loose the lottery.

Converse Barcan Formulas and Neighborhood Frames

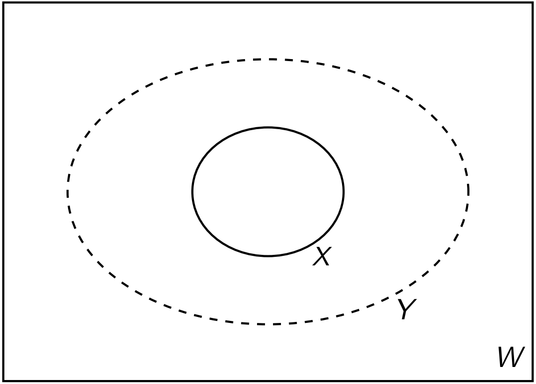
A frame $\mathcal{F}$ is **consistent** iff for each $w \in W$, $N(w) \neq \emptyset$

A first-order neighborhood frame $\mathcal{F} = \langle W, N, D \rangle$ is **nontrivial** iff $|D| > 1$

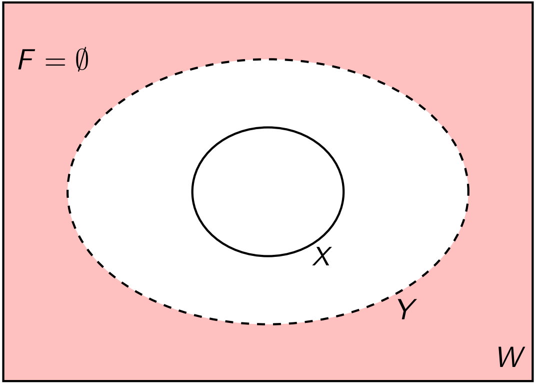
Lemma Let $\mathcal{F}$ be a consistent constant domain neighborhood frame. The converse Barcan formula is valid on $\mathcal{F}$ iff either $\mathcal{F}$ is trivial or $\mathcal{F}$ is supplemented.



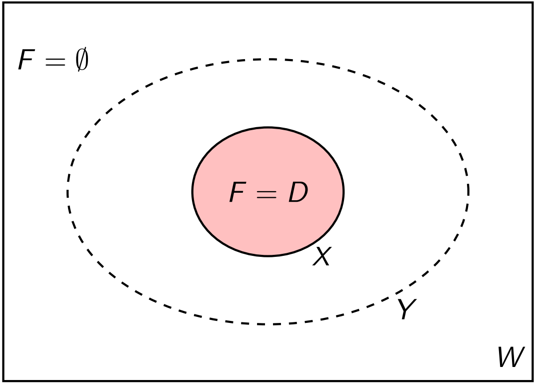
$$X \in N(w)$$



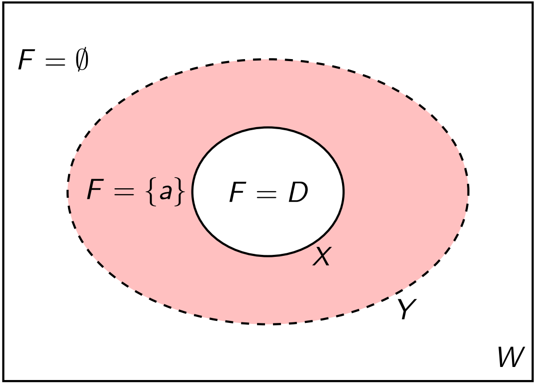
$Y \notin N(w)$



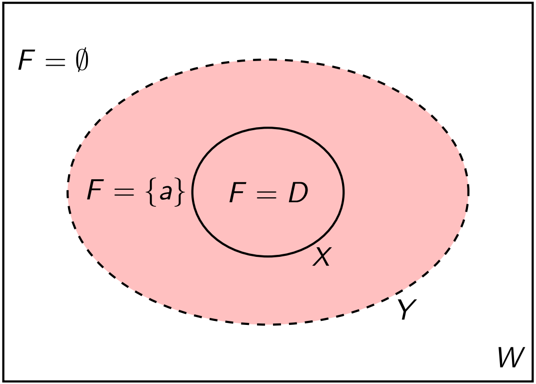
$$\forall v \notin Y, \;\; I(F, v) = \emptyset$$



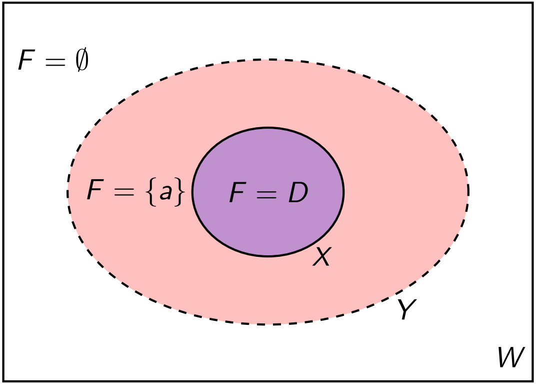
$$\forall v \in X, \quad I(F, v) = D = \{a, b\}$$



$$\forall v \in Y - X, \quad I(F, v) = D = \{a\}$$



$$(F[a])^{\mathcal{M}} = Y \not\in N(w) \quad \text{hence} \quad w \not\models \forall x \Box F(x)$$



$$(\forall x F(x))^{\mathcal{M}} = (F[a])^{\mathcal{M}} \cap (F[b])^{\mathcal{M}} = X \in N(w)$$

$$\text{hence} \quad w \models \Box \forall x F(x)$$

Barcan Formulas and Neighborhood Frames

We say that a frame closed under $\leq \kappa$ intersections if for each state w and each collection of sets $\{X_i \mid i \in I\}$ where $|I| \leq \kappa$, $\cap_{i \in I} X_i \in N(w)$.

Lemma Let $\mathcal{F}$ be a consistent constant domain neighborhood frame. The Barcan formula is valid on $\mathcal{F}$ iff either

1. $\mathcal{F}$ is trivial or
2. if D is finite, then $\mathcal{F}$ is closed under finite intersections and if D is infinite and of cardinality κ, then $\mathcal{F}$ is closed under $\leq \kappa$ intersections.

Suppose that **L** is a propositional modal logic. Let **FOL** + **L** denote the set of formulas closed under the following rules and axiom schemes

L All axiom schemes and rules from **L**.

(All) $\forall x\varphi(x) \rightarrow \varphi[y/x]$ is an axiom scheme, where y is free for x in φ.

(Gen) $\dfrac{\varphi \rightarrow \psi}{\varphi \rightarrow \forall x\psi}$, where x is not free in φ.

Theorem **FOL** $+$ **E** is sound and strongly complete with respect to the class of **all** constant domain neighborhood frames.

CBF

$$\vdash_{\mathbf{FOL+EM}} \Box\forall x\varphi(x) \rightarrow \forall x\Box\varphi(x)$$

$$\not\vdash_{\mathbf{FOL+E}+(\mathrm{CBF})} \Box(\varphi \wedge \psi) \rightarrow (\Box\varphi \wedge \Box\psi)$$

Completeness Theorems

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Completeness Theorems

Theorem FOL + E is sound and strongly complete with respect to the class of **all** frames.

Theorem FOL + EC is sound and strongly complete with respect to the class of frames that are closed under intersections.

Theorem FOL + EM is sound and strongly complete with respect to the class of supplemented frames.

Completeness Theorems

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Theorem $\mathbf{FOL} + \mathbf{EC}$ is sound and strongly complete with respect to the class of frames that are closed under intersections.

Theorem $\mathbf{FOL} + \mathbf{EM}$ is sound and strongly complete with respect to the class of supplemented frames.

Theorem $\mathbf{FOL} + \mathbf{E} + CBF$ is sound and strongly complete with respect to the class of frames that are either non-trivial and supplemented or trivial and not supplemented.

$\mathbf{FOL} + \mathbf{K}$ and $\mathbf{FOL} + \mathbf{K} + BF$

Theorem $\mathbf{FOL} + \mathbf{K}$ is sound and strongly complete with respect to the class of filters.

$\mathbf{FOL} + \mathbf{K}$ and $\mathbf{FOL} + \mathbf{K} + BF$

Theorem $\mathbf{FOL} + \mathbf{K}$ is sound and strongly complete with respect to the class of filters.

Observation The augmentation of the smallest canonical model for $\mathbf{FOL} + \mathbf{K}$ is not a canonical model for $\mathbf{FOL} + \mathbf{K}$. In fact, the closure under infinite intersection of the minimal canonical model for $\mathbf{FOL} + \mathbf{K}$ is not a canonical model for $\mathbf{FOL} + \mathbf{K}$.

$\mathbf{FOL} + \mathbf{K}$ and $\mathbf{FOL} + \mathbf{K} + BF$

Theorem $\mathbf{FOL} + \mathbf{K}$ is sound and strongly complete with respect to the class of filters.

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Lemma The augmentation of the smallest canonical model for $\mathbf{FOL} + \mathbf{K} + BF$ is a canonical for $\mathbf{FOL} + \mathbf{K} + BF$.

Theorem $\mathbf{FOL} + \mathbf{K} + BF$ is sound and strongly complete with respect to the class of augmented first-order neighborhood frames.

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1. **S4M** is complete for the class of all frames that are reflexive, transitive and *final* (every world can see an 'end-point'). However **FOL** + **S4M** is incomplete for Kripke models based on **S4M**-frames. (see Hughes and Cresswell, pg. 283).

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1. **S4M** is complete for the class of all frames that are reflexive, transitive and *final* (every world can see an 'end-point'). However **FOL** + **S4M** is incomplete for Kripke models based on **S4M**-frames. (see Hughes and Cresswell, pg. 283).

2. **S4**.**2** is S4 with $\Diamond\Box\varphi \rightarrow \Box\Diamond\varphi$. This logics is complete for the class of frames that are reflexive, transitive and *convergent*. However, **FOL** + **S4M** + *BF* is incomplete for the class of constant domain models based on reflexive, transitive and convergent frames. (see Hughes and Creswell, pg. 271)

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3. The quantified extension of **GL** is not complete (with respect to varying domains models).

What is going on?

R. Goldblatt. *Quantifiers, Propositions and Identity: Admissible Semantics for Quantified Modal and Substructural Logics*. Lecture Notes in Logic No. 38, Cambridge University Press, 2011.

Background: Incompleteness

There are (consistent) modal logics that are incomplete

A general model is a structure $\langle W, R, V, \mathcal{A}\rangle$ where $\mathcal{A}$ is a suitable boolean algebra with an operator of propositions.

All modal logics are sound and strongly complete with respect to general frames.

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- New perspective on the Barcan formula: it corresponds to **Tarskian models**
- There is a trade-off between having the underlying Kripke frame validate the propositional logic in question and having a Tarskian-reading of the quantifier.

Central Idea

Algebraic reading of the universal quantifier: $\forall x \varphi$ is true at a world w iff there is some proposition X such that X entails every instantiation of φ and X obtains at w.

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$\mathcal{M}, w \models_\sigma \forall x A$ iff there is a proposition X such that $w \in X$ and $X \subseteq (A)^{\mathcal{M}}_{\sigma(x|d)}$ for all $d \in D$.

vs.

$\mathcal{M}, w \models_\sigma \forall x A$ iff for all $d \in D$, $\mathcal{M}, w \models_{\sigma(x|d)} A$

General Frames

Let $\langle W, R \rangle$ be a frame.

$[R] : \wp W \to \wp W$ where
$[R](X) = \{w \in W \mid \text{ for all } v \in W, wRv \text{ implies } v \in X\}$
So $(\Box\alpha)^{\mathcal{M}} = [R](\alpha)^{\mathcal{M}}$

$X \Rightarrow Y = (W - X) \cup Y$
So $(\alpha \to \beta)^{\mathcal{M}} = (\alpha)^{\mathcal{M}} \Rightarrow (\beta)^{\mathcal{M}}$.

Halmos Functions

$\varphi : D^{\mathcal{V}} \to \wp W$
Let φ and ψ be two such functions, we can lift $[R]$ and $\Rightarrow$ to operations of functions: Eg., if $\varphi : D^{\mathcal{V}} \to \wp W$ and $f \in D^{\mathcal{V}}$.
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Fix a set $Prop \subseteq \wp W$. This defines for each $S \subseteq \wp W$,

$$\sqcap S = \bigcup \{X \in Prop \mid X \subseteq \bigcap S\}$$

General Frames for First-Order Modal Logic

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General Frames for First-Order Modal Logic

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$\langle W, R, V, Prop, PropFun \rangle$ where

- $Prop$ contains $\emptyset$ and is closed under $\Rightarrow$ and $[R]$
- Contains the function $\varphi_\emptyset(f) = \emptyset$ for all $f \in D^{\mathcal{V}}$
- PropFun is closed under $\Rightarrow$, $[R]$ and $\forall_x$.
- Assume $(P)^{\mathcal{M}} : D^{\mathcal{V}} \to \wp W$ is an element of $PropFun$ for each atomic predicate P.

General Completeness

Theorem For any propositional modal logic **S**, the quantified logic **QS** is complete for the class of (all validating) quantified general frames.

Note that the canonical model construction has as worlds maximally consistent sets that need not be $\forall$*-complete*.

Key Results

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Logics containing the Barcan formula have **two** characterizing canonical general frames: one that is Tarskian and one that is not.

1. If **S** is canonical, then the second canonical model will have an underlying propositional frame that validates **S** (eg., **S4.2**), but may not be Tarskian.
2. On the other hand, The Tarskian canonical model may not have an underlying propositional frame that is a frame for **S** (again **S4.2** is an example).

R. Goldblatt. *Quantifiers, Propositions and Identity: Admissible Semantics for Quantified Modal and Substructural Logics*. Lecture Notes in Logic No. 38, Cambridge University Press and the Association for Symbolic Logic, 2011.

An Application: Coalition Logic

G. Boella, D. Gabbay, V. Genovese, L. van der Torre. *Higher-Order Coalition Logic*. 19th European Conference on Artificial Intelligence, pgs. 555 - 560, 2010.

Q. Chen and K. Su. *Higher-Order Epistemic Coalition Logic for Multi-Agent Systems*. 7th Workshop on Logical Aspects of Multi-Agent Systems, 2014.

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Higher-Order Coalition Logic: $\varphi :=$
$F(x_1, \ldots, x_n) \mid Xx \mid \neg\varphi \mid \varphi \wedge \varphi \mid \forall X\varphi \mid \forall x\varphi \mid [\{x\}\varphi]\varphi \mid \langle\{x\}\varphi\rangle\varphi$

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- $F(x_1, \ldots, x_n)$ is a first-order atomic formula
- x is a first-order variable
- X is a set variable
- $\{x\}\psi$ is a group operator representing the set of all d such that $\psi[d/x]$ holds

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- Complex relationships between coalitions and agents:
  $[\{x\}\varphi(x)]\psi \rightarrow [\{y\}\exists x(\varphi(x) \wedge collaborates(y,x))]\psi$

  If the coalition represented by φ can achieve ψ then so can any group that collaborates with at least one member of $\varphi(x)$.

HCL: Barcan/Converse Barcan Formulas

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For each person at ILLC, I can make them happy does not imply that I can do something to make everyone at ILLC happy.

Higher-Order Coalition Logic

Sound and complete axiomatization combines ideas from coalitional logic, first-order extensions of non-normal modal logics and Henkin-style completeness for second-order logic.

Richer Languages

✓ Normal + non-normal modalities

✓ First-order extensions

- Fixed-point operators/group notions (group evidence, common belief)
- Dynamic extensions (game logic, updating neighborhood models, evidence dynamics)