Neighborhood Semantics for Modal Logic
Lecture 5

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Text

Neighborhood semantics for modal logic (Draft)

Ch 1: Introduction and Motivation

Ch 2: Core Theory: Expressivity, Completeness, Decidability, Complexity, Correspondence Theory

Ch 3: Richer Languages: Fixed-point operators, First-order extensions, Dynamic operators

Schedule

Lecture 1: June 1st, 14h00-16h30

Lecture 2: June 2nd 12h30-14h30

Lecture 3: June 7th, 14h00-16h30

Lecture 4: June 8th, 11h00-13h00

Lecture 5: June 8th, 14h00-16h30

Lecture 7: June 9th, 12h30-14h30

Lecture 8: June 13th,12h30-15h00

Lecture 9: June 14th, 10h00-13h00

Lecture 10 Presentations (solutions to problems etc.): June 15th, 10h00-13h00

Frame Correspondence

Definition

A modal formula φ defines a property P of neighborhood functions if any neighborhood frame $\mathfrak{F}$ has property P iff $\mathfrak{F}$ validates φ.

What can we say?

Lemma

Let $\mathfrak{F} = \langle W, N \rangle$ *be a neighborhood frame. Then* $\mathfrak{F} \models \Box(\varphi \wedge \psi) \rightarrow \Box\varphi \wedge \Box\psi$ *iff* $\mathfrak{F}$ *is closed under supersets.*

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Lemma

Let $\mathfrak{F} = \langle W, N \rangle$ *be a neighborhood frame. Then* $\mathfrak{F} \models \Box\varphi \wedge \Box\psi \rightarrow \Box(\varphi \wedge \psi)$ *iff* $\mathfrak{F}$ *is closed under finite intersections.*

What can we say?

Consider the formulas $\Diamond\top$ and $\Box\varphi \rightarrow \Diamond\varphi$.

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On relational frames, these formulas both define the same property: seriality.

On neighborhood frames:

- $\Diamond\top$ corresponds to the property $\emptyset \notin N(w)$
- $\Box\varphi \rightarrow \Diamond\varphi$ is valid on $\mathfrak{F}$ iff $\mathfrak{F}$ is proper.

What can we say?

Lemma

Let $\mathfrak{F} = \langle W, N \rangle$ be a neighborhood frame such that for each $w \in W$, $N(w) \neq \emptyset$.

1. $\mathfrak{F} \models \Box\varphi \rightarrow \varphi$ *iff for each* $w \in W$, $w \in \cap N(w)$
2. $\mathfrak{F} \models \Box\varphi \rightarrow \Box\Box\varphi$ *iff for each* $w \in W$, *if* $X \in N(w)$, *then* $\{v \mid X \in N(v)\} \in N(w)$

Find properties on frames that are defined by the following formulas:

1. $\Box\bot$
2. $\neg\Box\varphi \to \Box\neg\Box\varphi$
3. $\Diamond\varphi \to \Box\varphi$
4. $\Diamond\Box\varphi \to \Box\Diamond\varphi$
5. $\Box\Diamond\varphi \to \Diamond\Box\varphi$

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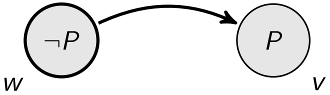
What about more general results (e.g., Sahlqvist Theorem, Standard Translation, van Benthem Characterization Theorem, etc.?)

- From neighborhoods to orders
- Simulating non-normal modal logics

Beliefs via Plausibility



Epistemic-Plausibility Model: $\mathcal{M} = \langle W, \preceq, V\rangle$

- $w \preceq v$ means w is at least as plausibility as v. ($\preceq$ is reflexive, transitive, connected, well-founded)

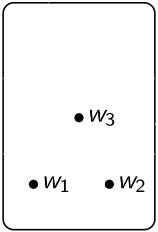
Language: $\varphi := p \mid \neg\varphi \mid \varphi \wedge \psi \mid B^{\varphi}\psi \mid [\preceq]\varphi \mid A\varphi$

Truth:

- $Max_{\preceq}(X) = \{w \in X \mid \text{there is no } v \in X \text{ such that } w \prec v\}$
- $[\![\varphi]\!]_{\mathcal{M}} = \{w \mid \mathcal{M}, w \models \varphi\}$
- $\mathcal{M}, w \models B^{\varphi}\psi$ iff for all $v \in Max_{\preceq}([\![\varphi]\!]_{\mathcal{M}})$, $\mathcal{M}, v \models \psi$

Beliefs via Plausibility

$W = \{w_1, w_2, w_3\}$



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$w_2 \preceq w_1$ and $w_1 \preceq w_2$ (w_1 and w_2 are equi-plausbile)

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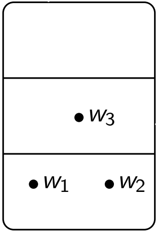

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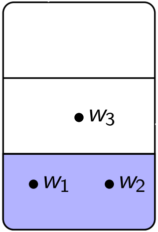
$\{w_1, w_2\} \subseteq Max_{\preceq}(W)$



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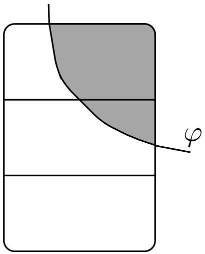

Conditional Belief: $B^{\varphi}\psi$

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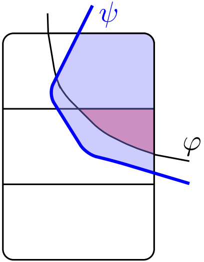

Conditional Belief: $B^{\varphi}\psi$

$$Max_{\preceq}([\![\varphi]\!]_{\mathcal{M}}) \subseteq [\![\psi]\!]_{\mathcal{M}}$$

Evidence Models and Plausibility Models

What is the precise relationship between evidence models and plausibility models?

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Three issues

1. Plausibility orders that are not *connected*
2. Conditional beliefs on evidence models
3. From evidence to plausibility (and back)

Incomparability in Plausibility Models

In general, we must drop the assumption that $\preceq$ is connected.

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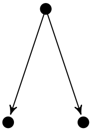
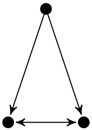
Incomparability arises as the result of receiving incompatible evidence:

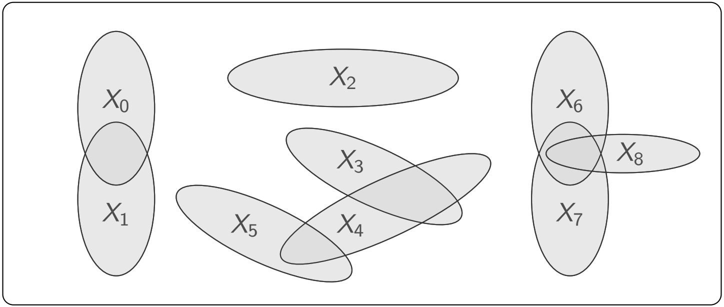


Incompatible evidence



Compatible evidence

Conditional Beliefs on Evidence Models



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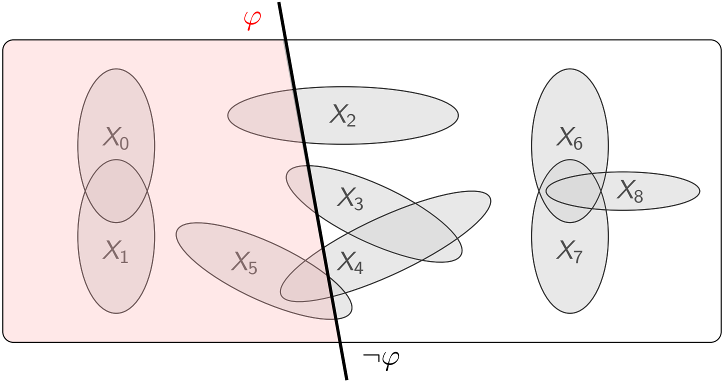

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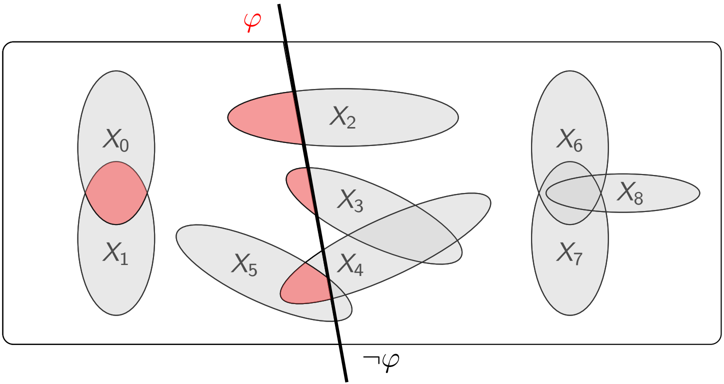

Conditional Beliefs on Evidence Models

$B^{\varphi}\psi$: "the agent believes ψ conditional on φ."

Main idea: Ignore the evidence that is inconsistent with φ.

Relativized *w*-**scenario**: Suppose that $X \subseteq W$. Given a collection $\mathcal{X} \subseteq \wp(W)$, let $\mathcal{X}^X = \{Y \cap X \mid Y \in \mathcal{X}\}$. We say that a collection $\mathcal{X}$ of subsets of W has the **finite intersection property relative to** X (X-f.i.p.) if, $\mathcal{X}^X$ as the f.i.p. and is maximal if $\mathcal{X}^X$ is.

- $\mathcal{M}, w \models B^{\varphi}\psi$ iff for each maximal φ-f.i.p. $\mathcal{X} \subseteq E(w)$, for each $v \in \bigcap \mathcal{X}^{\varphi}$, $\mathcal{M}, v \models \psi$

Conditional Beliefs: Example

$B\psi \rightarrow B^{\varphi}\psi$ is not valid.

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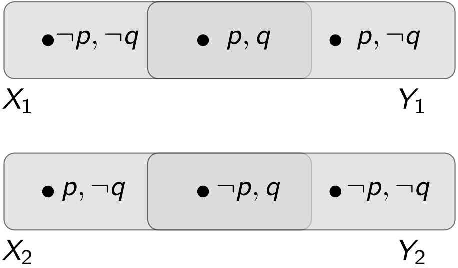

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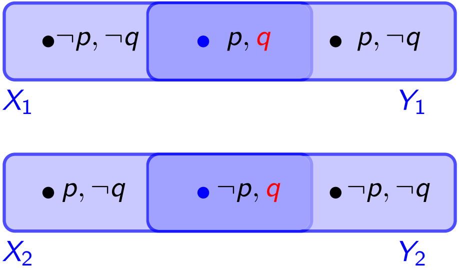

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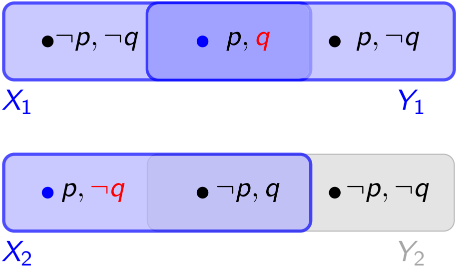

✓ $\mathcal{M}, w \models Bq$

- $\mathcal{M}, w \not\models B^{p} q$

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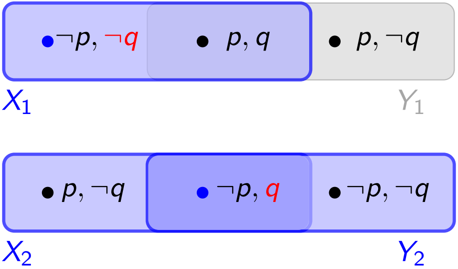

- ✓ $\mathcal{M}, w \models Bq$
- ✓ $\mathcal{M}, w \not\models B^{p}q$
- ▸ $\mathcal{M}, w \not\models B^{\neg p}q$

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- $\mathcal{M}, w \models \langle\ ]^{\varphi}\psi$ iff there exists an evidence set $X \in E(w)$ consistent with φ such that for all $v \in X \cap [\![\varphi]\!]_{\mathcal{M}}$, $\mathcal{M}, v \models \psi$.

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$\langle\ ]^{\varphi}\psi$ is not equivalent to $\langle\ ](\varphi \rightarrow \psi)$: if there is no evidence consistent with φ, then $\langle\ ]^{\varphi}\psi$ is false.

Plausibility Models $\hookrightarrow$ Evidence Models

Let $\mathcal{M} = \langle W, \preceq, V \rangle$ be a plausibility model.

The evidence sets are the upwards $\preceq$-closed sets of worlds.

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- Given a $X \subseteq W$, let $X{\uparrow_{\preceq}} = \{v \in W \mid \exists x \in X \text{ and } x \preceq v\}$
- A set $X \subseteq W$ is $\preceq$-closed if $X{\uparrow} \subseteq X$.

Evidence model generated from $\mathcal{M}$: $EV(\mathcal{M}) = \langle W, \mathcal{E}^{\preceq}, V\rangle$
with $\mathcal{E}^{\preceq} = \{X \mid \emptyset \neq X \text{ is } \preceq\text{-closed }\}$

$\mathcal{L}(\preceq, B, A)$ is generated by

$$p \mid \neg\varphi \mid \varphi \wedge \psi \mid [B]\psi \mid [\preceq]\varphi \mid [A]\varphi$$

Suppose that $\mathcal{M} = \langle W, \preceq, V\rangle$ is a plausibility model.

- $\mathcal{M}, w \models [B]\varphi$ iff $Max_{\preceq}(W) \subseteq [\![\varphi]\!]_{\mathcal{M}}$
- $\mathcal{M}, w \models [\preceq]\varphi$ iff for all $v \in W$, if $w \preceq v$ then $\mathcal{M}, v \models \varphi$
- $\mathcal{M}, w \models [A]\varphi$ iff for all $v \in W$, $\mathcal{M}, v \models \varphi$.

On finite plausibility models, the belief modality $[B]$ is definable in terms of the $[A]$ and $[\preceq]$ modalities:

$$B\varphi \;:=\; [A]\langle\preceq\rangle[\preceq]\varphi$$

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The translation $tr_{\preceq} : \mathcal{L}(\langle\ ], A) \rightarrow \mathcal{L}([\preceq], A)$ is defined as follows:

- for each $p \in \mathrm{At}$, $tr_{\preceq}(p) = p$;
- $tr_{\preceq}(\neg\varphi) = \neg tr_{\preceq}(\varphi)$ and $tr_{\preceq}(\varphi \wedge \psi) = tr_{\preceq}(\varphi) \wedge tr_{\preceq}(\psi)$;
- $tr_{\preceq}([A]\varphi) = [A](tr_{\preceq}(\varphi))$; and
- $tr_{\preceq}(\langle\ ]\varphi) = \langle E\rangle[\preceq](tr_{\preceq}(\varphi))$.

Proposition. Let $\mathcal{M} = \langle W, \preceq, V \rangle$ be a plausibility model. For any $\varphi \in \mathcal{L}(\langle\ ], A)$ and state $w \in W$,

$$\mathcal{M}, w \models tr_{\preceq}(\varphi) \text{ iff } \mathcal{M}^{\preceq}, w \models \varphi.$$

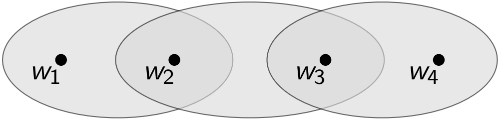
Evidence Models $\hookrightarrow$ Plausibility Models

Specialization Order

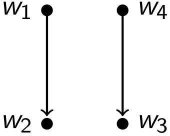
Suppose that $\langle W, \mathcal{F} \rangle$ is a subset space. Define $\preceq_{\mathcal{F}} \subseteq W \times W$ as follows:

$$w \preceq_{\mathcal{F}} v \text{ iff for all } X \in \mathcal{F}, \text{ if } w \in X, \text{ then } v \in X$$

Specialization Order: Example



$\mathcal{F}$



$\preceq_{\mathcal{F}}$

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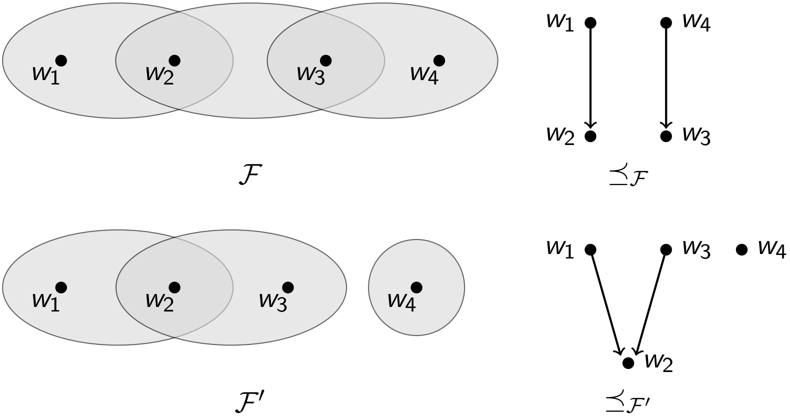

$\mathfrak{M} = \langle W, N, V \rangle$. For each $w \in W$, define a plausibility ordering $\preceq_{N(w)}$.

Taking Stock

Language $\varphi \;:=\; p \mid \neg\varphi \mid \varphi \wedge \psi \mid \langle\ ]\varphi \mid [B]\varphi \mid [A]\varphi \mid [\preceq]\varphi$

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General Model: $\langle W, E, R_B, \preceq, V\rangle$ where

1. for each $w \in W$, $\varnothing \notin E(w)$ and $W \in E(w)$;
2. for all $w, v, u \in W$, if $w \preceq v$ and $w \in X \in E(u)$, then $v \in X$;
3. for all w, v, u, if $w \preceq v$ and $u \, R_B \, v$ then $u \, R_B \, w$.

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3. for all w, v, u, if $w \preceq v$ and $u\, R_B\, v$ then $u\, R_B\, w$.

Intended Model: $\langle W, E, V\rangle \hookrightarrow \langle W, E, R_B^E, \preceq^E, V\rangle$ where

1. $w\, R_B^E\, v$ iff $v \in \cap\mathcal{X}$ for some w-scenario $\mathcal{X}$
2. $w \preceq^E v$ iff whenever u, X are such that $w \in X \in E(u)$, then $v \in X$

Given an evidence model $\mathfrak{M} = \langle W, E, V \rangle$, define the extended model

$$\mathfrak{M}^{\triangle} = \langle W, E, B_E, \preceq_E, V \rangle.$$

where

- $w \; B_E \; v$ iff $v \in \bigcap \mathcal{X}$ for some w-scenario $\mathcal{X}$, and
- $\preceq_E = \preceq_{\cup_{w \in W} E(w)}$ (i.e., $w \preceq_E v$ iff for any u, X, if $w \in X \in E(u)$, then $v \in X$).

Say that $\mathcal{M}$ is an **intended model** provided $\mathcal{M} = \langle W, E, V \rangle^{\triangle}$

What is the precise relationship between intended models $\mathfrak{M}^{\triangle}$ and extended evidence models $\mathcal{M} = \langle W, E, B, \preceq, V \rangle$.

Lemma. Suppose that $\mathfrak{M} = \langle W, E, V \rangle$ is an evidence model, then $\mathfrak{M}^{\triangle}$ is a model according to the above definition.

Lemma. If $\mathcal{M} = \langle W, \mathcal{E}, B_{\mathcal{E}}, \preceq_{\mathcal{E}}, V \rangle$ is uniform and intended, then for every scenario $\mathcal{X}$ and every $w \in \bigcap \mathcal{X}$, w is $\preceq$-maximal if and only if w lies in $\bigcap \mathcal{X}'$ for some scenario $\mathcal{X}'$.
Moreover, if $\mathcal{M}$ is flat then the sets of the form $\bigcap \mathcal{X}$ with $\mathcal{X}$ a scenario are precisely the $\preceq_{\mathcal{E}}$-equivalence classes of maximal worlds.

The plausibility orders in extended evidence models satisfy an additional property:

Let $\preceq$ be a plausibility order over W. Say $D \subseteq W$ is **directed** if any two elements of D have an upper bound in D.

A plausibility order $\preceq$ satisfies the **boundendess condition** if every directed set D has an upper bound (not necessarily in D).

Proposition. If an evidence model is flat, then its derived plausibility relation satisfies the boundedness condition.

Lemma. If $\mathcal{M}$ is flat and $\preceq_E$ is its derived plausibility relation, then for every w there is v such that $w \preceq_E v$ and v is maximal.

Theorem. Over the class of uniform evidence models with derived plausibility relation, $[A]\langle\preceq\rangle[\preceq]\varphi \rightarrow [B]\varphi$ is valid.

Over the class of models that are moreover flat, the two formulas are equivalent.

Suppose that $\langle W, \mathcal{F} \rangle$ is a subset space. Define $\preceq_{\mathcal{F}} \subseteq W \times W$ as follows:

$$w \preceq_{\mathcal{F}} v \text{ iff for all } X \in \mathcal{F}, \text{ if } w \in X, \text{ then } v \in X$$

Two ways to generalize:

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1. $\mathcal{F}_w = \{X \in \mathcal{F} \mid w \in X\}$

   $w \preceq_{\mathcal{F}} v$ iff $\mathcal{F}_w \leq \mathcal{F}_v$

2. A set of *reasons* $\mathcal{R} \subseteq \mathcal{F}$ may be associated with arbitrary orderings: $\mathcal{R} \mapsto \preceq_{\mathcal{R}}$.

F. Dietrich and C. List. *Reasons for (prior) belief in Bayesian epistemology*.
.

Let $\mathcal{D} \subseteq \wp(W)$ be a set of *doxastic reasons*.

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Let $\mathbb{D}$ be the space of doxastic reasons. Assume that $\mathbb{D}$ is closed under finite intersections and finite unions.

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Possible meeting points (**Doxastic Possibilities**):

Union station (u)
Lincoln Memorial (l)
White House (w)
Eric's House in Chevy Chase (e)

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Two NASSLLI participants need to meet in Washington DC at noon tomorrow, but they did not settle on a location.

Possible meeting points (**Doxastic Possibilities**):

Union station (u)
Lincoln Memorial (l)
White House (w)
Eric's House in Chevy Chase (e)

Doxastic Reasons

$A = \{u\}$: The place in question is where one arrives in DC
$F = \{l, w\}$: The place in question is world-famous
$H = \{w, e\}$: A family lives at the place in question

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$$\mathcal{D} = \{A, F, H\}: \quad u \succ_{\mathcal{D}} l \succ_{\mathcal{D}} w \succ_{\mathcal{D}} e$$
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Can we find a **credibility ordering** $\geq$ on the space of doxastic reasons $\mathbb{D}$ such that

$$w \succeq_{\mathcal{D}} v \text{ iff } \{R \mid w \in R \in \mathcal{D}\} \geq \{R \mid v \in R \in \mathcal{D}\}?$$

Axiom 1 (Principle of insufficient reason): For any $w, v \in W$ and any $\mathcal{D} \in \mathbb{D}$

$$\text{if } \{R \mid w \in R \in \mathcal{D}\} = \{R \mid v \in R \in \mathcal{D}\}, \text{ then } w \sim_{\mathcal{D}} v$$

Axiom 2: For any $w, v \in W$ and any $\mathcal{D}_2, \mathcal{D}_2 \in \mathbb{D}$ with $\mathcal{D}_1 \subseteq \mathcal{D}_2$,

$$\text{if, [for all } R \in \mathcal{D}_2 - \mathcal{D}_1,\ w, v \notin R], \text{ then } [w \succeq_{\mathcal{D}_1} v \Leftrightarrow w \succeq_{\mathcal{D}_2} v]$$

Theorem (Dietrich and List). The agent's plausibility orderings $(\succeq_{\mathcal{D}})_{\mathcal{D} \in \mathbb{D}}$ satisfies Axiom 1 and Axiom 2 if and only if there is a credibility ordering $\geq$ on $\mathbb{D}$ such that for all $\mathcal{D} \in \mathbb{D}$,

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for all $w, v \in W$.

✓ From neighborhoods to orders

- Simulating non-normal modal logics

We can *simulate* any non-normal modal logic with a bi-modal normal modal logic.

Definition

Given a neighborhood model $\mathcal{M} = \langle W, N, V \rangle$, define a Kripke model $\mathcal{M}^{\circ} = \langle V, R_N, R_{\not\ni}, R_N, Pt, V \rangle$ as follows:

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Let $\mathcal{L}'$ be the language

$$\varphi := p \mid \neg\varphi \mid \varphi \wedge \psi \mid [\ni]\varphi \mid [\not\ni]\varphi \mid [N]\varphi \mid \mathrm{Pt}$$

where $p \in \mathrm{At}$ and Pt is a unary modal operator.

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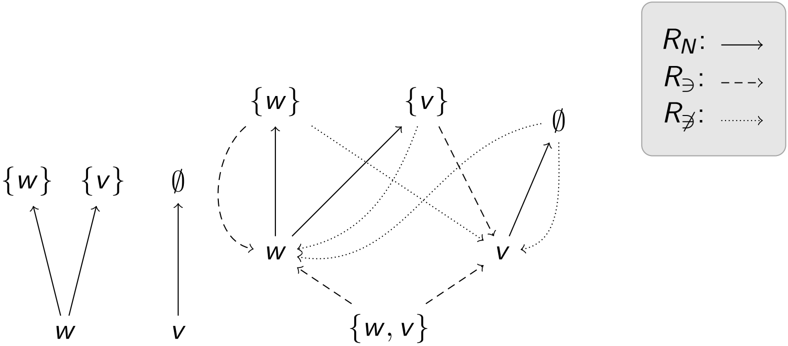
Lemma

For each neighborhood model $\mathcal{M} = \langle W, N, V \rangle$ and each formula $\varphi \in \mathcal{L}$, for any $w \in W$,

$$\mathcal{M}, w \models \varphi \text{ iff } \mathcal{M}^\circ, w \models ST(\varphi)$$



$\mathfrak{M}$ $\qquad\qquad$ $\mathfrak{M}^\circ$

$\mathfrak{M}, w \models \Box p$ and $\mathfrak{M}, v \models \Box\bot$.

- $\mathfrak{M}^\circ, w \models \langle N\rangle([\ni]p \wedge [\not\ni]\neg p)$ and $\mathfrak{M}^\circ, v \not\models \langle N\rangle([\ni]p \wedge [\not\ni]\neg p)$
- $\mathfrak{M}^\circ, v \models \langle N\rangle([\ni]\bot \wedge [\not\ni]\top)$ and $\mathfrak{M}^\circ, w \not\models \langle N\rangle([\ni]\bot \wedge [\not\ni]\top)$

Monotonic Models

Lemma

On Monotonic Models $\langle N\rangle([\ni]ST(\varphi) \wedge [\not\ni]\neg ST(\varphi))$ *is equivalent to* $\langle N\rangle([\ni]ST(\varphi))$

O. Gasquet and A. Herzig. *From Classical to Normal Modal Logic*. in Proof Theory of Modal Logic, Kluwer, pgs. 293 - 311, 1996.

M. Kracht and F. Wolter. *Normal Monomodal Logics can Simulate all Others*. The Journal of Symbolic Logic, 64:1, pgs. 99 - 138, 1999.

Disjoint Union

Let $\mathbb{M}_1 = \langle W_1, N_1, V_1 \rangle$ and $\mathbb{M}_2 = \langle W_2, N_2, V_2 \rangle$ be two neighborhood models. The **disjoint union of** $\mathbb{M}_1$ **and** $\mathfrak{M}_2$ is the neighborhood model $\mathbb{M}_1 + \mathbb{M}_2 = \langle W_1 + W_2, N, V \rangle$ where for all $p \in \mathsf{At}$, $V(p) = V_1(p) \cup V_2(p)$; and for $i = 1, 2$,

for all $X \subseteq W_1 + W_2$, and $w \in W_i$, $X \in N(w)$ iff $X \cap W_i \in N_i(w)$.

(Similar definition for frames)

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for all $X \subseteq W_1 + W_2$, and $w \in W_i$, $X \in N(w)$ iff $X \cap W_i \in N_i(w)$.

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Proposition. For all $\varphi \in \mathcal{L}$, for $i = 1, 2$, if $w \in W_i$, then $\mathfrak{M}_1 + \mathfrak{M}_2, w \models \varphi$ iff $\mathfrak{M}_i, w \models \varphi$.

Fact. The universal modality is not definable in the basic modal language.

Monotonic Bisimulation

Let $\mathfrak{M} = \langle W, N, V \rangle$ and $\mathfrak{M}' = \langle W', N', V' \rangle$ be two monotonic neighborhood models. A relation $Z \subseteq W \times W'$ is a bisimulation provided whenever wZw':

Atomic harmony: for each $p \in \mathsf{At}$, $w \in V(p)$ iff $w' \in V'(p)$

Zig: If $X \in N(w)$ then there is an $X' \subseteq W'$ such that

$$X' \in N'(w') \text{ and } \forall x' \in X' \ \exists x \in X \text{ such that } xZx'$$

Zag: If $X' \in N'(w')$ then there is an $X \subseteq W$ such that

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Lemma

On locally core-finite models, if $\mathbb{M}, w \equiv_{\mathcal{L}} \mathbb{M}', w'$ *then* $\mathbb{M}, w \underline{\leftrightarrow} \mathbb{M}', w'$.

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Theorem

A first-order formula (in the appropriate language...) $\alpha(x)$ is invariant for monotonic bisimulation, then $\alpha(x)$ is equivalent to $st_x^{mon}(\varphi)$ for some $\varphi \in \mathcal{L}$.

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M. Pauly. *Bisimulation for Non-normal Modal Logic*. 1999.

H. Hansen. *Monotonic Modal Logic*. 2003.

Do monotonic bisimulations work when we drop monotonicity? No!

Bounded Morphisms

If $\mathbb{M}_1 = \langle W_1, N_1, V_1 \rangle$ and $\mathbb{M}_2 = \langle W_2, N_2, V_2 \rangle$ are two neighborhood models, and $f : W_1 \to W_2$ is a function, then f is a **(frame) bounded morphism** if

for all $X \subseteq W_2$, we have $f^{-1}[X] \in N_1(w)$ iff $X \in N_2(f(w))$;

and for all $p \in \text{At}$, and all $w \in W_1$: $w \in V_1(p)$ iff $f(s) \in V_2(p)$.

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Lemma Let $\mathbb{M}_1 = \langle W_1, N_1, V_1\rangle$ and $\mathbb{M}_2 = \langle W_2, N_2, V_2\rangle$ be two neighborhood models and $f : \mathbb{M}_1 \to \mathbb{M}_2$ a bounded morphism. For each modal formula $\varphi \in \mathcal{L}$ and state $w \in W_1$, $\mathbb{M}_1, w \models \varphi$ iff $\mathbb{M}_2, f(w) \models \varphi$.

Definition

Two points w_1 from $\mathfrak{F}_1$ and w_2 from $\mathfrak{F}_2$ are behavorially equivalent provided there is a neighborhood frame $\mathfrak{F}$ and bounded morphisms $f : \mathfrak{F}_1 \to \mathfrak{F}$ and $g : \mathfrak{F}_2 \to \mathfrak{F}$ such that $f(w_1) = g(w_2)$.

Theorem

Over the class **N** *(of neighborhood models), the following are equivalent:*

- $\alpha(x)$ *is equivalent to the translation of a modal formula*
- $\alpha(x)$ *is invariant under behavioural equivalence.*

H. Hansen, C. Kupke and EP. *Neighbourhood Structures: Bisimilarity and Basic Model Theory*. Logical Methods in Computer Science, 5(2:2), pgs. 1 - 38, 2009.

The Language $\mathcal{L}_2$

The language $\mathcal{L}_2$ is built from the following grammar:

$$x = y \mid u = v \mid \mathsf{P}_i x \mid x\mathsf{N}u \mid u\mathsf{E}x \mid \neg\varphi \mid \varphi \wedge \psi \mid \exists x\varphi \mid \exists u\varphi$$

$\mathfrak{M} = \langle D, \{P_i \mid i \in \omega\}, N, E\rangle$ where

- $D = D^{\mathsf{s}} \cup D^{\mathsf{n}}$ (and $D^{\mathsf{s}} \cap D^{\mathsf{n}} = \emptyset$),
- $Q_i \subseteq D^{\mathsf{s}}$,
- $N \subseteq D^{\mathsf{s}} \times D^{\mathsf{n}}$ and
- $E \subseteq D^{\mathsf{n}} \times D^{\mathsf{s}}$.

The Language $\mathcal{L}_2$

Definition

Let $\mathfrak{M} = \langle S, N, V \rangle$ be a neighbourhood model. The *first-order translation* of $\mathcal{M}$ is the structure $\mathfrak{M}^\circ = \langle D, \{P_i \mid i \in \omega\}, R_N, R_\ni \rangle$ where

- $D^{\mathsf{s}} = S$, $D^{\mathsf{n}} = \bigcup_{s \in S} N(s)$
- For each $i \in \omega$, $P_i = V(p_i)$
- $R_N = \{(s, U) \mid s \in D^{\mathsf{s}}, U \in N(s)\}$
- $R_\ni = \{(U, s) \mid s \in D^{\mathsf{s}}, s \in U\}$

The Language $\mathcal{L}_2$

Definition

The *standard translation* of the basic modal language are functions $st_x : \mathcal{L} \to \mathcal{L}_2$ defined as follows as follows: $st_x(p_i) = \mathsf{P}_i x$, st_x commutes with boolean connectives and

$$st_x(\Box\varphi) = \exists u(x\mathsf{R}_N u \wedge (\forall y(u\mathsf{R}_\ni y \leftrightarrow st_y(\varphi)))$$

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Lemma

Let $\mathfrak{M}$ be a neighbourhood structure and $\varphi \in \mathcal{L}$. For each $s \in S$, $\mathfrak{M}, s \models \varphi$ iff $\mathfrak{M}^\circ \models st_x(\varphi)[s]$.

$\mathbf{N} = \{\mathfrak{M} \mid \mathfrak{M} \cong \mathfrak{M}^\circ \text{ for some neighbourhood model } \mathfrak{M}\}$

(*A*1) $\exists x(x = x)$

(*A*2) $\forall u \exists x(x \mathsf{R}_N u)$

(*A*3)

$$\forall u, v(\neg(u = v) \to \exists x((u \mathsf{R}_\ni x \wedge \neg v \mathsf{R}_\ni x) \vee (\neg u \mathsf{R}_\ni x \wedge v \mathsf{R}_\ni x)))$$

Theorem

Suppose $\mathfrak{M}$ is an $\mathcal{L}_2$-structure. Then there is a neighbourhood structure $\mathfrak{M}_\circ$ such that $\mathfrak{M} \cong (\mathfrak{M}_\circ)^\circ$.

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