Neighborhood Semantics for Modal Logic
Lecture 4

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Text

Neighborhood semantics for modal logic (Draft)

Ch 1: Introduction and Motivation

Ch 2: Core Theory: Expressivity, Completeness, Decidability, Complexity, Correspondence Theory

Ch 3: Richer Languages: Fixed-point operators, First-order extensions, Dynamic operators

Schedule

Lecture 1: June 1st, 14h00-16h30

Lecture 2: June 2nd 12h30-14h30

Lecture 3: June 7th, 14h00-16h30

Lecture 4: June 8th, 11h00-13h00

Lecture 5: June 8th, 14h00-16h30

Lecture 7: June 9th, 12h30-14h30

Lecture 8: June 13th,12h30-15h00

Lecture 9: June 14th, 10h00-13h00

Lecture 10 Presentations (solutions to problems etc.): June 15th, 10h00-13h00

Core Theory

- ✓ Neighborhood Semantics in the Broader Logical Landscape
- ✓ Bisimulations
- Completeness, Decidability, Complexity
- Incompleteness
- Relation with Relational Semantics
- Model Theory

Suppose that Γ is a set of formulas and F is a class of neighborhood frames. A formula $\varphi \in \mathcal{L}$ is a **semantic consequence** with respect to F of Γ, denoted $\Gamma \models_{\mathsf{F}} \varphi$, provided for each model $\mathfrak{M} = \langle W, N, V\rangle$ based on a frame from F (i.e., $\langle W, N\rangle \in \mathsf{F}$), for each $w \in W$, if $\mathcal{M}, w \models \Gamma$, then $\mathcal{M}, w \models \varphi$.

Some Notation

- A formula $\varphi \in \mathcal{L}$ is valid in F ($\models_{\mathsf{F}} \varphi$) if for each $\mathfrak{F} \in \mathsf{F}$, $\mathfrak{F} \models \varphi$.

- We say that a logic **L** is sound with respect to F, provided $\vdash_{\mathbf{L}} \varphi$ implies $\models_{\mathsf{F}} \varphi$.

- A logic **L** is weakly complete with respect to a class of frames F, if $\models_{\mathsf{F}} \varphi$ implies $\vdash_{\mathbf{L}} \varphi$.

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A set of formulas Γ is called a **maximally consistent set** provided Γ is a consistent set of formulas and for all formulas $\varphi \in \mathcal{L}$, either $\varphi \in \Gamma$ or $\neg\varphi \in \Gamma$.

Let $M_{\mathbf{L}}$ be the set of **L**-maximally consistent sets of formulas.

The **L**-proof set of $\varphi \in \mathcal{L}$ is $|\varphi|_{\mathbf{L}} = \{\Gamma \mid \varphi \in \Gamma\}$.

Let **L** be a logic and $\varphi, \psi \in \mathcal{L}$. Then

1. $|\varphi \wedge \psi|_{\mathbf{L}} = |\varphi|_{\mathbf{L}} \cap |\psi|_{\mathbf{L}}$
2. $|\neg\varphi|_{\mathbf{L}} = M_{\mathbf{L}} - |\varphi|_{\mathbf{L}}$
3. $|\varphi \vee \psi|_{\mathbf{L}} = |\varphi|_{\mathbf{L}} \cup |\psi|_{\mathbf{L}}$
4. $|\varphi|_{\mathbf{L}} \subseteq |\psi|_{\mathbf{L}}$ iff $\vdash_{\mathbf{L}} \varphi \rightarrow \psi$
5. $|\varphi|_{\mathbf{L}} = |\psi|_{\mathbf{L}}$ iff $\vdash_{\mathbf{L}} \varphi \leftrightarrow \psi$
6. For any maximally **L**-consistent set Γ, if $\varphi \in \Gamma$ and $\varphi \rightarrow \psi \in \Gamma$, then $\psi \in \Gamma$
7. For any maximally **L**-consistent set Γ, If $\vdash_{\mathbf{L}} \varphi$, then $\varphi \in \Gamma$

Lindenbaum's Lemma. For any consistent set of formulas Γ, there exists a maximally consistent set Γ' such that $\Gamma \subseteq \Gamma'$.

Canonical Model

Definition

A neighborhood model $\mathfrak{M} = \langle W, N, V \rangle$ is canonical for **L** provided

- $W = \{$ maximally **L**-consistent sets $\}$

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- for all $\varphi \in \mathcal{L}$ and $\Gamma \in W$, $|\varphi|_{\mathbf{L}} \in N(\Gamma)$ iff $\Box\varphi \in \Gamma$
- for all $p \in \mathrm{At}$, $V(p) = |p|_{\mathbf{L}}$

Examples of Canonical Models

$\mathfrak{M}_{\mathbf{L}}^{min} = \langle M_{\mathbf{L}}, N_{\mathbf{L}}^{min}, V_{\mathbf{L}} \rangle$, where for each $\Gamma \in M_{\mathbf{L}}$,
$N_{\mathbf{L}}^{min}(\Gamma) = \{|\varphi|_{\mathbf{L}} \mid \Box\varphi \in \Gamma\}$.

Examples of Canonical Models

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Let $P_{\mathbf{L}} = \{|\varphi|_{\mathbf{L}} \mid \varphi \in \mathcal{L}\}$ be the set of all proof sets.

$\mathfrak{M}_{\mathbf{L}}^{max} = \langle M_{\mathbf{L}}, N_{\mathbf{L}}^{max}, V_{\mathbf{L}} \rangle$, where for each $\Gamma \in M_{\mathbf{L}}$,
$N_{\mathbf{L}}^{max}(\Gamma) = N_{\mathbf{L}}^{min}(\Gamma) \cup \{X \mid X \subseteq M_{\mathbf{L}}, X \not\in P_{\mathbf{L}}\}$

The canonical model works...

Lemma

For any logic $\mathbf{L}$ *containing the rule RE, if* $N_{\mathbf{L}} : M_{\mathbf{L}} \to \wp(\wp(M_{\mathbf{L}}))$ *is a function such that for each* $\Gamma \in M_{\mathbf{L}}$, $|\varphi|_{\mathbf{L}} \in N_{\mathbf{L}}(\Gamma)$ *iff* $\Box\varphi \in \Gamma$. *Then if* $|\varphi|_{\mathbf{L}} \in N_{\mathbf{L}}(\Gamma)$ *and* $|\varphi|_{\mathbf{L}} = |\psi|_{\mathbf{L}}$, *then* $\Box\psi \in \Gamma$.

Lemma (Truth Lemma)

For any consistent classical modal logic $\mathbf{L}$ *and any consistent formula* φ, *if* $\mathfrak{M}$ *is canonical for* $\mathbf{L}$,

$$[\![\varphi]\!]_{\mathfrak{M}} = |\varphi|_{\mathbf{L}}$$

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Lemma (Truth Lemma)

For any consistent classical modal logic **L** *and any consistent formula* φ, *if* $\mathfrak{M}$ *is canonical for* **L**,

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The Proofs

Theorem

The logic **E** *is sound and strongly complete with respect to the class of all neighborhood frames.*

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Lemma

If $C \in \mathbf{L}$, *then* $\langle M_{\mathbf{L}}, N_{\mathbf{L}}^{min} \rangle$ *is closed under finite intersections.*

Theorem

The logic **EC** *is sound and strongly complete with respect to the class of neighborhood frames that are closed under intersections.*

The Proofs

Fact: $\langle M_{\mathbf{EM}}, N^{min}_{\mathbf{EM}} \rangle$ is not closed under supersets.

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Lemma

Suppose that $\mathfrak{M} = sup(\mathfrak{M}^{min}_{\mathbf{EM}})$. *Then* $\mathfrak{M}$ *is canonical for* **EM**.

Theorem

The logic **EM** *is sound and strongly complete with respect to the class of supplemented frames.*

The Proofs

Theorem

The logic **K** *is sound and strongly complete with respect to the class of filters.*

Theorem

The logic **K** *is sound and strongly complete with respect to the class of augmented frames.*

The Normal Situation

The smallest normal modal logic **K** consists of

PC Your favorite axioms of **PC**

K $\Box(\varphi \to \psi) \to \Box\varphi \to \Box\psi$

Nec $\dfrac{\vdash \varphi}{\Box\varphi}$

MP $\dfrac{\vdash \varphi \to \psi \qquad \vdash \varphi}{\psi}$

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Theorem: K is sound and strongly complete with respect to the class of all Kripke frames.

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Theorem: For all $\Gamma \subseteq \mathcal{L}$, $\Gamma \vdash_{\mathbf{K}} \varphi$ iff $\Gamma \models \varphi$.

The Normal Situation

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MP $\dfrac{\vdash \varphi \to \psi \qquad \vdash \varphi}{\psi}$

Theorem: **K** $+ \Box\varphi \to \varphi + \Box\varphi \to \Box\Box\varphi$ is sound and strongly complete with respect to the class of all reflexive and transitive Kripke frames.

A logic **L** is **neighborhood complete** (resp. **Kripke complete**) provided there is a class of neighborhood frames F (resp. relational frames) such that $\mathbf{L} = \mathbf{L}(\mathsf{F}) = \{\varphi \in \mathcal{L} \mid \mathfrak{F} \models \varphi \text{ for all } \mathfrak{F} \in \mathsf{F}\}$. Otherwise, the logic is said to be **neighborhood incomplete** (resp. **Kripke incomplete**).

Incompleteness

There are (consistent) modal logics that are incomplete:

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Theorem Let **TMEQ** be the following normal modal logic:

- **K**
- $\Box\varphi \to \varphi$
- $\Box\Diamond\varphi \to \Diamond\Box\varphi$
- $\Diamond(\Diamond\varphi \wedge \Box\psi) \to \Box(\Diamond\varphi \vee \Box\varphi)$
- $(\Diamond\varphi \wedge \Box(\varphi \to \Box\varphi)) \to \varphi$

There is no class of frames validating precisely the formulas in **TMEQ**.

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There is no class of frames validating precisely the formulas in **TMEQ**.

J. van Benthem. *Two Simple Incomplete Modal Logics*. Theoria (1978).

Incompleteness?

Are all modal logics complete with respect to some class of neighborhood frames?

Incompleteness?

Are all modal logics complete with respect to some class of neighborhood frames? No

Incompleteness

Martin Gerson. *The Inadequacy of Neighbourhood Semantics for Modal Logic*. Journal of Symbolic Logic (1975).

There are two logics $\mathbf{L}$ and $\mathbf{L}'$ that are incomplete with respect to neighborhood semantics.

Incompleteness

Martin Gerson. *The Inadequacy of Neighbourhood Semantics for Modal Logic*. Journal of Symbolic Logic (1975).

There are two logics **L** and $\mathbf{L}'$ that are incomplete with respect to neighborhood semantics.

(there are formulas φ and φ' that are valid in the class of frames for **L** and $\mathbf{L}'$ respectively, but φ and φ' are not deducible in the respective logics).

Incompleteness

Martin Gerson. *The Inadequacy of Neighbourhood Semantics for Modal Logic*. Journal of Symbolic Logic (1975).

There are two logics **L** and **L**′ that are incomplete with respect to neighborhood semantics.

L is between **T** and **S4**

L′ is above **S4** (adapts Fine's incomplete logic)

$$
\begin{aligned}
A_i &= \Box(q_i \to r) \qquad (i = 1, 2)\\
B_i &= \Box(r \to \Diamond q_i) \qquad (i = 1, 2)\\
C_1 &= \Box\neg(q_1 \wedge q_2)\\
A &= r \wedge \Box p \wedge \neg\Box\Box p \wedge A_1 \wedge A_2 \wedge B_1 \wedge B_2 \wedge\\
&\quad\; C_1 \to \Diamond(r \wedge \Box(r \to (q_1 \vee q_2))\\
D &= (p \wedge \Diamond\Diamond q) \to (\Diamond q \vee \Diamond\Diamond(q \wedge \Diamond p))\\
E &= (\Box p \wedge \neg\Box\Box p) \to \Diamond(\Box\Box p \wedge \neg\Box\Box\Box p)\\
F &= \Box p \to \Box\Box p
\end{aligned}
$$

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A &= r \wedge \Box p \wedge \neg\Box\Box p \wedge A_1 \wedge A_2 \wedge B_1 \wedge B_2 \wedge\\
&\quad C_1 \rightarrow \Diamond(r \wedge \Box(r \rightarrow (q_1 \vee q_2))\\
D &= (p \wedge \Diamond\Diamond q) \rightarrow (\Diamond q \vee \Diamond\Diamond(q \wedge \Diamond p))\\
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\end{aligned}$$

Let **L** be the logic obtained by adding A, D, and E as additional axioms to **T**.

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F &= \Box p \to \Box\Box p
\end{aligned}
$$

Let **L** be the logic obtained by adding A, D, and E as additional axioms to **T**.

Theorem. (Gerson) The formula F is valid in all neighborhood frames for **L**, but it is not provable in **L**.

Comparing Relational and Neighborhood Semantics

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Fact: If a (normal) modal logic is complete with respect to some class of relational frames then it is complete with respect to some class of neighborhood frames.

What about the converse?

Are there normal modal logics that are incomplete with respect to relational semantics, but complete with respect to neighborhood semantics?

Comparing Relational and Neighborhood Semantics

Fact: If a (normal) modal logic is complete with respect to some class of relational frames then it is complete with respect to some class of neighborhood frames.

What about the converse?

Are there normal modal logics that are incomplete with respect to relational semantics, but complete with respect to neighborhood semantics? Yes!

Comparing Relational and Neighborhood Semantics

Neighborhood completeness does not imply Kripke completeness

- extension of **K**

D. Gabbay. *A normal logic that is complete for neighborhood frames but not for Kripke frames*. Theoria (1975).

- extension of **T**

M. Gerson. *A Neighbourhood frame for T with no equivalent relational frame*. Zeitschr. J. Math. Logik und Grundlagen (1976).

- extension of **S4**

M. Gerson. *An Extension of S4 Complete for the Neighbourhood Semantics but Incomplete for the Relational Semantics*. Studia Logica (1975).

Let **Gab** be the smallest set of formulas that contains all instances of axioms and the rules from the minimal normal modal logic **K** plus the following four axiom schemes:

$$A1 \quad (\Box\psi \wedge \neg\Box\Box\psi) \rightarrow (\Box\Box\varphi \rightarrow \Box\varphi)$$

$$A2 \quad (\Box\psi \wedge \neg\Box\Box\psi) \rightarrow \Box((\Diamond\Box\varphi \wedge \Box(\Box\varphi \rightarrow \varphi)) \rightarrow \Box\varphi)$$

$$A3 \quad (\Box\psi \wedge \neg\Box\Box\psi) \rightarrow (\neg\Box\varphi \leftrightarrow \Box\neg\varphi)$$

$$A4 \quad (\Box\psi \wedge \neg\Box\Box\psi) \rightarrow \Box[(\Box\varphi \rightarrow \Box\Box\varphi) \wedge \Box(\Box\varphi \rightarrow \Box\Box\varphi)]$$

Claim. There is a logic **Gab*** that is complete with respect to some class of neighborhood frames, but incomplete with respect to any class of relational frames (Theorem **??**).

The proof uses a special neighborhood model based on a frame $\mathfrak{F}_{Gab} = \langle W, R_G, \mathcal{U} \rangle$ with

- $W = \mathbb{N} \times \mathbb{Z}$ (where $\mathbb{N}$ is the set of natural numbers and $\mathbb{Z}$ is the set of integers);
- $R_G \subseteq W \times W$ be defined as follows:

  $$(m, n)\ R_G\ (m', n') \text{ iff } n < n' \text{ or } n = n' \text{ and } m > m'; \text{ and}$$
- $\mathcal{U}$ is a non-principal ultrafilter on $\mathbb{N}$.

Suppose that $\mathfrak{M} = \langle W, R_G, \mathcal{U}, V\rangle$ is a model based on $\mathfrak{F}_{Gab}$, where $V : \mathsf{At} \rightarrow \wp(W)$ is a valuation function. Truth of the basic modal language is defined as follows:

- $\mathfrak{M}, (m, n) \models p$ iff $(m, n) \in V(p)$
- $\mathfrak{M}, (m, n) \models \neg\varphi$ iff $\mathfrak{M}, (m, n) \not\models \varphi$
- $\mathfrak{M}, (m, n) \models \varphi \wedge \psi$ iff $\mathfrak{M}, (m, n) \models \varphi$ and $\mathfrak{M}, (m, n) \models \psi$
- $\mathfrak{M}, (m, n) \models \Box\varphi$ iff
  - If $m \neq 0$, then for all (m', n'), if $(m, n)\ R_G\ (m', n')$, then $\mathfrak{M}, (m', n') \models \varphi$
  - If $m = 0$, then $\{y \mid \mathfrak{M}, (y, n+1) \models \varphi\} \in \mathcal{U}$

$\mathbf{Gab}^* = \{\varphi \mid \mathfrak{F}_{Gab} \models \varphi\}$

$\alpha = \Box p \wedge \neg\Box\Box p \wedge \Box\Diamond(\neg q \wedge \Box q)$

1. $\neg\alpha \notin \mathbf{Gab}^*$
2. Any class of relational frames that validate **Gab** must also validate $\neg\alpha$.
3. $\mathfrak{F}_{Gab}$ validates *Gab*

The general situation is not very well understood.

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Notable exceptions:

L. Chagrova. *On the Degree of Neighborhood Incompleteness of Normal Modal Logics*. AiML 1 (1998).

V. Shehtman. *On Strong Neighbourhood Completeness of Modal and Intermediate Propositional Logics (Part I)*. AiML 1 (1998).

T. Litak. *Modal Incompleteness Revisited*. Studia Logica (2004).

Recovering Completeness

Definition

A general neighborhood frame is a tuple $\mathfrak{F}^g = \langle W, N, \mathcal{A}\rangle$, where $\langle W, N\rangle$ is a neighborhood frame and $\mathcal{A}$ is a collection of subsets of W closed under intersections, complements, and the m_N operator.

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A valuation $V : \text{At} \rightarrow \wp(W)$ is admissible for a general frame $\langle W, N, \mathcal{A} \rangle$ if for each $p \in \text{At}$, $V(p) \in \mathcal{A}$.

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Definition

Suppose that $\mathfrak{F}^g = \langle W, N, \mathcal{A} \rangle$ is a general neighborhood frame. A general modal based on $\mathfrak{F}^g$ is a tuple $\mathfrak{M}^g = \langle W, N, \mathcal{A}, V \rangle$ where V is an admissible valuation.

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Lemma

Let $\mathfrak{M}^g$ be an general neighborhood model. Then for each $\varphi \in \mathcal{L}$, $[\![\varphi]\!]_{\mathfrak{M}^g} \in \mathcal{A}$.

Recovering Completeness

Definition

A general neighborhood frame is a tuple $\mathfrak{F}^g = \langle W, N, \mathcal{A} \rangle$, where $\langle W, N \rangle$ is a neighborhood frame and $\mathcal{A}$ is a collection of subsets of W closed under intersections, complements, and the m_N operator.

Lemma

Let **L** *be any logic extending* **E**. *Then the general canonical frame validates* **L** *($\mathfrak{F}^g_{\mathbf{L}} \models \mathbf{L}$).*

Corollary

Any modal logic extending **E** *is strongly complete with respect to some class of general frames.*

Summary

For any modal logic **L**:

- If **L** is Kripke complete, then it is neighborhood complete
- **L** is complete with respect to its class of general frames

Summary

For any modal logic **L**:

- If **L** is Kripke complete, then it is neighborhood complete
- **L** is complete with respect to its class of general frames

There are modal logics showing that

- neighborhood completeness does not imply Kripke completeness
- algebraic completeness does not imply neighborhood completeness

Core Theory

- ✓ Neighborhood Semantics in the Broader Logical Landscape
- ✓ Bisimulations
- Completeness, Decidability, Complexity
- Incompleteness
- Relation with Relational Semantics
- Model Theory

Filtrations

Let $\mathfrak{M} = \langle W, N, V \rangle$ be a neighborhood model and suppose that Σ is a set of sentences from $\mathcal{L}$.

For each $w, v \in W$, we say $w \sim_\Sigma v$ iff for each $\varphi \in \Sigma$, $w \models \varphi$ iff $v \models \varphi$.

For each $w \in W$, let $[w]_\Sigma = \{v \mid w \sim_\Sigma v\}$ be the equivalence class of $\sim_\Sigma$.

If $X \subseteq W$, let $[X]_\Sigma = \{[w] \mid w \in X\}$.

Filtrations

Definition

Let $\mathfrak{M} = \langle W, N, V\rangle$ be a neighborhood model and Σ a set of sentences closed under subformulas. A filtration of $\mathfrak{M}$ through Σ is a model $\mathfrak{M}^f = \langle W^f, N^f, V^f\rangle$ where

1. $W^f = [W]$
2. For each $w \in W$
   2.1 for each $\Box\varphi \in \Sigma$, $[\![\varphi]\!]_{\mathbb{M}} \in N(w)$ iff $[[\![\varphi]\!]_{\mathbb{M}}] \in N^f([w])$
3. For each $p \in \text{At}$, $V(p) = [V(p)]$

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3. For each $p \in \mathrm{At}$, $V(p) = [V(p)]$

Theorem

Suppose that $\mathfrak{M}^f = \langle W^f, N^f, V^f\rangle$ is a filtration of $\mathfrak{M} = \langle W, N, V\rangle$ through (a subformula closed) set of sentences Σ. Then for each $\varphi \in \Sigma$, $\mathfrak{M}, w \models \varphi$ iff $\mathfrak{M}^f, [w] \models \varphi$

Filtrations

Definition

Let $\mathfrak{M} = \langle W, N, V\rangle$ be a neighborhood model and Σ a set of sentences closed under subformulas. A filtration of $\mathfrak{M}$ through Σ is a model $\mathfrak{M}^f = \langle W^f, N^f, V^f\rangle$ where

1. $W^f = [W]$
2. For each $w \in W$
   2.1 for each $\Box\varphi \in \Sigma$, $[\![\varphi]\!]_{\mathbb{M}} \in N(w)$ iff $[[\![\varphi]\!]_{\mathbb{M}}] \in N^f([w])$
3. For each $p \in \mathrm{At}$, $V(p) = [V(p)]$

Corollary

E *has the finite model property. I.e., if φ has a model then there is a finite model.*

A Few Comments on Complexity

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