Neighborhood Semantics for Modal Logic
Lecture 3

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Text

Neighborhood semantics for modal logic (Draft)

Ch 1: Introduction and Motivation

Ch 2: Core Theory: Expressivity, Completeness, Decidability, Complexity, Correspondence Theory

Ch 3: Richer Languages: Fixed-point operators, First-order extensions, Dynamic operators

Schedule

Lecture 1: June 1st, 14h00-16h30

Lecture 2: June 2nd 12h30-14h30

Lecture 3: June 7th, 14h00-16h30

Lecture 4: June 8th, 11h00-13h00

Lecture 5: June 8th, 14h00-16h30

Lecture 7: June 9th, 12h30-14h30

Lecture 8: June 13th,12h30-15h00

Lecture 9: June 14th, 10h00-13h00

Lecture 10 Presentations (solutions to problems etc.): June 15th, 10h00-13h00

1. Non-normal modal logics
2. Neighborhood semantics for modal logic

Non-normal modal logics

(M)	$\Box(\varphi \wedge \psi) \rightarrow \Box\varphi \wedge \Box\psi$
(C)	$\Box\varphi \wedge \Box\psi \rightarrow \Box(\varphi \wedge \psi)$
(N)	$\Box\top$
(K)	$\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$
(Dual)	$\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$
(Nec)	from $\vdash \varphi$ infer $\vdash \Box\varphi$
(Re)	from $\vdash \varphi \leftrightarrow \psi$ infer $\vdash \Box\varphi \leftrightarrow \Box\psi$

Non-normal modal logics

~~(M)~~ ~~$\Box(\varphi \wedge \psi) \rightarrow \Box\varphi \wedge \Box\psi$~~

~~(C)~~ ~~$\Box\varphi \wedge \Box\psi \rightarrow \Box(\varphi \wedge \psi)$~~

~~(N)~~ ~~$\Box\top$~~

~~(K)~~ ~~$\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$~~

(Dual) $\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$

~~(Nec)~~ ~~from $\vdash \varphi$ infer $\vdash \Box\varphi$~~

(Re) from $\vdash \varphi \leftrightarrow \psi$ infer $\vdash \Box\varphi \leftrightarrow \Box\psi$

PC Propositional Calculus

E $\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$

M $\Box(\varphi \wedge \psi) \rightarrow (\Box\varphi \wedge \Box\psi)$

C $(\Box\varphi \wedge \Box\psi) \rightarrow \Box(\varphi \wedge \psi)$

N $\Box\top$

K $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$

RE $\dfrac{\varphi \leftrightarrow \psi}{\Box\varphi \leftrightarrow \Box\psi}$

Nec $\dfrac{\varphi}{\Box\varphi}$

MP $\dfrac{\varphi \quad \varphi \rightarrow \psi}{\psi}$

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A modal logic **L** is classical if it contains all instances of *E* and is closed under *RE*.

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E is the smallest classical modal logic.

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E is the smallest classical modal logic.

In **E**, *M* is equivalent to

$$(Mon)\ \frac{\varphi \rightarrow \psi}{\Box\varphi \rightarrow \Box\psi}$$

PC Propositional Calculus

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EM is the logic **E** + *Mon*

PC 6. Propositional Calculus

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E is the smallest classical modal logic.

EM is the logic **E** + *Mon*

EC is the logic **E** + *C*

PC Propositional Calculus

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EMC is the smallest regular modal logic

PC Propositional Calculus

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A logic is normal if it contains all instances of *E*, *C* and is closed under *Mon* and *Nec*

PC Propositional Calculus

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E is the smallest classical modal logic.

EM is the logic **E** + *Mon*

EC is the logic **E** + *C*

EMC is the smallest regular modal logic

K is the smallest normal modal logic

PC Propositional Calculus

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E is the smallest classical modal logic.

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EC is the logic **E** + *C*

EMC is the smallest regular modal logic

K = **EMCN**

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EMC is the smallest regular modal logic

K $= PC(+E) + K + Nec + MP$

Useful Fact

Theorem (Uniform Substitution)

The following rule can be derived in **E**

$$\frac{\psi \leftrightarrow \psi'}{\varphi \leftrightarrow \varphi[\psi/\psi']}$$

Interesting Fact

Each of K, M and C are logically independent:

- $\mathbf{EC} \nvdash K$
- $\mathbf{EM} \nvdash K$
- $\mathbf{EMC} \vdash K$
- $\mathbf{EK} \nvdash M$
- $\mathbf{EK} \nvdash C$

Neighborhood Frames

Let W be a non-empty set of states.

Any function $N : W \to \wp(\wp(W))$ is called a neighborhood function

A pair $\langle W, N\rangle$ is a called a neighborhood frame if W a non-empty set and N is a neighborhood function.

A neighborhood model based on $\mathfrak{F} = \langle W, N\rangle$ is a tuple $\langle W, N, V\rangle$ where $V : \text{At} \to \wp(W)$ is a valuation function.

Truth in a Model

- $\mathfrak{M}, w \models p$ iff $w \in V(p)$
- $\mathfrak{M}, w \models \neg\varphi$ iff $\mathfrak{M}, w \not\models \varphi$
- $\mathfrak{M}, w \models \varphi \wedge \psi$ iff $\mathfrak{M}, w \models \varphi$ and $\mathfrak{M}, w \models \psi$

Truth in a Model

- $\mathfrak{M}, w \models p$ iff $w \in V(p)$
- $\mathfrak{M}, w \models \neg\varphi$ iff $\mathfrak{M}, w \not\models \varphi$
- $\mathfrak{M}, w \models \varphi \wedge \psi$ iff $\mathfrak{M}, w \models \varphi$ and $\mathfrak{M}, w \models \psi$
- $\mathfrak{M}, w \models \Box\varphi$ iff $[\![\varphi]\!]_{\mathfrak{M}} \in N(w)$
- $\mathfrak{M}, w \models \Diamond\varphi$ iff $W - [\![\varphi]\!]_{\mathfrak{M}} \not\in N(w)$

where $[\![\varphi]\!]_{\mathfrak{M}} = \{w \mid \mathfrak{M}, w \models \varphi\}$.

Let $N : W \to \wp\wp W$ be a neighborhood function and define $m_N : \wp W \to \wp W$:

$$\text{for } X \subseteq W,\ m_N(X) = \{w \mid X \in N(w)\}$$

1. $[\![p]\!]_{\mathfrak{M}} = V(p)$ for $p \in \mathsf{At}$
2. $[\![\neg\varphi]\!]_{\mathfrak{M}} = W - [\![\varphi]\!]_{\mathfrak{M}}$
3. $[\![\varphi \wedge \psi]\!]_{\mathfrak{M}} = [\![\varphi]\!]_{\mathfrak{M}} \cap [\![\psi]\!]_{\mathfrak{M}}$
4. $[\![\Box\varphi]\!]_{\mathfrak{M}} = m_N([\![\varphi]\!]_{\mathfrak{M}})$
5. $[\![\Diamond\varphi]\!]_{\mathfrak{M}} = W - m_N(W - [\![\varphi]\!]_{\mathfrak{M}})$

Why non-normal modal logic? ✓

Why neighborhood models? ✓

- ✓ Logical omniscience
- ✓ Logics of knowledge and beliefs
- ✓ Logic of high probability
- ✓ Logics of classical deduction
- ✓ Deontic logics
- ✓ Logics of ability
- ✓ Logics of group decision making
- ???

- ✓ Subset spaces, neighborhood frames/models, reasoning about subset spaces
- ✓ Logic of knowledge, evidence and belief
- ✓ Coalitional logic
- Interesting mathematical structures: Ultrafilters, topologies, ✓ hypergraphs

The Broader Logical Landscape

- Relational Models
- Topological Models
- n-ary Relational Structures
- Plausibility Structures
- First-Order Logic

From Kripke Frames to Neighborhood Frames

Let $R \subseteq W \times W$, define a map $R^{\rightarrow} : W \rightarrow \wp W$:

$$\text{for each } w \in W, \text{ let } R^{\rightarrow}(w) = \{v \mid wRv\}$$

From Kripke Frames to Neighborhood Frames

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Definition

Given a relation R on a set W and a state $w \in W$. A set $X \subseteq W$ is *R-necessary at w* if $R^{\rightarrow}(w) \subseteq X$.

From Kripke Frames to Neighborhood Frames

Let $R \subseteq W \times W$, define a map $R^{\rightarrow} : W \rightarrow \wp W$:

$$\text{for each } w \in W, \text{ let } R^{\rightarrow}(w) = \{v \mid wRv\}$$

Let $\mathcal{N}_w^R$ be the set of sets that are R-necessary at w:

$$\mathcal{N}_w^R = \{X \mid R^{\rightarrow}(w) \subseteq X\}$$

From Kripke Frames to Neighborhood Frames

Let $R \subseteq W \times W$, define a map $R^{\rightarrow} : W \rightarrow \wp W$:

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Let $\mathcal{N}_w^R$ be the set of sets that are R-necessary at w:

$$\mathcal{N}_w^R = \{X \mid R^{\rightarrow}(w) \subseteq X\}$$

Lemma

Let R be a relation on W. Then for each $w \in W$, $\mathcal{N}_w^R$ is augmented.

From Kripke Frames to Neighborhood Frames

Properties of R are reflected in $\mathcal{N}_w^R$:

- If R is reflexive, then for each $w \in W$, $w \in \cap\mathcal{N}_w$
- If R is transitive then for each $w \in W$, if $X \in \mathcal{N}_w$, then $\{v \mid X \in \mathcal{N}_v\} \in \mathcal{N}_w$.

From Neighborhood Frames to Kripke Frames

Theorem

- *Let* $\langle W, R\rangle$ *be a relational frame. Then there is an equivalent augmented neighborhood frame.*
- *Let* $\langle W, N\rangle$ *be an augmented neighborhood frame. Then there is an equivalent relational frame.*

From Neighborhood Frames to Kripke Frames

for all $X \subseteq W$, $X \in N(w)$ iff $X \in \mathcal{N}_w^R$.

Theorem

- *Let $\langle W, R \rangle$ be a relational frame. Then there is an equivalent augmented neighborhood frame.*

- *Let $\langle W, N \rangle$ be an augmented neighborhood frame. Then there is an equivalent relational frame.*

From Neighborhood Frames to Kripke Frames

Theorem

✓ *Let $\langle W, R\rangle$ be a relational frame. Then there is an equivalent augmented neighborhood frame.*

- *Let $\langle W, N\rangle$ be an augmented neighborhood frame. Then there is an equivalent relational frame.*

Proof.

For each $w \in W$, let $N(w) = \mathcal{N}_w^R$. □

From Neighborhood Frames to Kripke Frames

Theorem

- *Let $\langle W, R \rangle$ be a relational frame. Then there is an equivalent augmented neighborhood frame.*

✓ *Let $\langle W, N \rangle$ be an augmented neighborhood frame. Then there is an equivalent relational frame.*

Proof.

For each $w, v \in W$, $wR_N v$ iff $v \in \cap N(w)$. □

Topological Models for Modal Logic

Definition

Topological Space A **topological space** is a neighborhood frame $\langle W, \mathcal{T} \rangle$ where W is a nonempty set and

1. $W \in \mathcal{T}$, $\emptyset \in W$
2. $\mathcal{T}$ is closed under finite intersections
3. $\mathcal{T}$ is closed under arbitrary unions.

Topological Models for Modal Logic

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2. $\mathcal{T}$ is closed under finite intersections
3. $\mathcal{T}$ is closed under arbitrary unions.

A neighborhood of w is any set X such that there is an $O \in \mathcal{T}$ with $w \in O \subseteq N$

Let $\mathcal{T}_w$ be the collection of all neighborhoods of w.

Topological Models for Modal Logic

Definition

Topological Space A **topological space** is a neighborhood frame $\langle W, \mathcal{T} \rangle$ where W is a nonempty set and

1. $W \in \mathcal{T}$, $\emptyset \in W$
2. $\mathcal{T}$ is closed under finite intersections
3. $\mathcal{T}$ is closed under arbitrary unions.

Lemma

Let $\langle W, \mathcal{T} \rangle$ be a topological space. Then for each $w \in W$, the collection $\mathcal{T}_w$ contains W, is closed under finite intersections and closed under arbitrary unions.

Topological Models for Modal Logic

The largest open subset of X is called the interior of X, denoted $Int(X)$. Formally,

$$Int(X) = \cup\{O \mid O \in \mathcal{T} \text{ and } O \subseteq X\}$$

The smallest closed set containing X is called the closure of X, denoted $Cl(X)$. Formally,

$$Cl(X) = \cap\{C \mid W - C \in \mathcal{T} \text{ and } X \subseteq C\}$$

Topological Models for Modal Logic

- $Int(X) = \cup\{O \mid O \in \mathcal{T} \text{ and } O \subseteq X\}$
- $Cl(X) = \cap\{C \mid W - C \in \mathcal{T} \text{ and } X \subseteq C\}$

Lemma

Let $\langle W, \mathcal{T}\rangle$ be a topological space and $X \subseteq W$. Then

1. $Int(X \cap Y) = Int(X) \cap Int(Y)$
2. $Int(\emptyset) = \emptyset$, $Int(W) = W$
3. $Int(X) \subseteq X$
4. $Int(Int(X)) = Int(X)$
5. $Int(X) = W - Cl(W - X)$

Topological Models for Modal Logic

- $Int(X) = \cup\{O \mid O \in \mathcal{T} \text{ and } O \subseteq X\}$
- $Cl(X) = \cap\{C \mid W - C \in \mathcal{T} \text{ and } X \subseteq C\}$

Lemma

Let $\langle W, \mathcal{T}\rangle$ be a topological space and $X \subseteq W$. Then

1. $\Box(\varphi \wedge \psi) \leftrightarrow \Box\varphi \wedge \Box\psi$
2. $\Box\bot \leftrightarrow \bot, \Box\top \leftrightarrow \top$
3. $\Box\varphi \rightarrow \varphi$
4. $\Box\Box\varphi \leftrightarrow \Box\varphi$
5. $\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$

Topological Models for Modal Logic

A topological model is a triple $\langle W, \mathcal{T}, V\rangle$ where $\langle W, \mathcal{T}\rangle$ is a topological space and V a valuation function.

Topological Models for Modal Logic

A topological model is a triple $\langle W, \mathcal{T}, V\rangle$ where $\langle W, \mathcal{T}\rangle$ is a topological space and V a valuation function.

$$\mathfrak{M}^T, w \models \Box\varphi \text{ iff } \exists O \in \mathcal{T}, w \in O \text{ such that } \forall v \in O, \mathfrak{M}^T, v \models \varphi$$

$$(\Box\varphi)^{\mathfrak{M}^T} = Int((\varphi)^{\mathfrak{M}^T})$$

From Neighborhoods to Topologies

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A family $\mathcal{B}$ of subsets of W is called a basis for a topology $\mathcal{T}$ if every open set can be represented as the union of elements of a subset of $\mathcal{B}$

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A family $\mathcal{B}$ of subsets of W is called a basis for a topology $\mathcal{T}$ if every open set can be represented as the union of elements of a subset of $\mathcal{B}$

Fact: A family $\mathcal{B}$ of subsets of W is a basis for some topology if

- for each $w \in W$ there is a $U \in \mathcal{B}$ such that $w \in U$
- for each $U, V \in \mathcal{B}$, if $w \in U \cap V$ then there is a $W \in \mathcal{B}$ such that $w \in W \subseteq U \cap V$

From Neighborhoods to Topologies

A family $\mathcal{B}$ of subsets of W is called a basis for a topology $\mathcal{T}$ if every open set can be represented as the union of elements of a subset of $\mathcal{B}$

Let $\mathbb{M} = \langle W, N, V \rangle$ be a neighborhood models. Suppose that N satisfies the following properties

- for each $w \in W$, $N(w)$ is a filter
- for each $w \in W$, $w \in \cap N(w)$
- for each $w \in W$ and $X \subseteq W$, if $X \in N(w)$, then $m_N(X) \in N(w)$

Then there is a topological model that is point-wise equivalent to $\mathbb{M}$.

J. van Benthem and G. Bezhanishvili. *Modal Logics of Space*. Handbook of Spatial Logics, pgs. 217 - 298, 2007.

Generalized Relational Models

- *n*-ary relations
- multiple relations
- non-normal worlds

n-ary Relations

$$(\Box\varphi \wedge \Box\psi) \rightarrow \Box(\varphi \wedge \psi)$$

n-ary Relations

$$(\Box\varphi \wedge \Box\psi) \rightarrow \Box(\varphi \wedge \psi)$$

An n-**ary relational model** is a tuple $\langle W, R, V\rangle$ where W is a non-empty set and $R \subseteq W^n$ is an n-ary relation ($R \subseteq W^n$) and $V : \mathsf{At} \rightarrow \wp(W)$ is a valuation function. (Assume $n \geq 2$)

n-ary Relations

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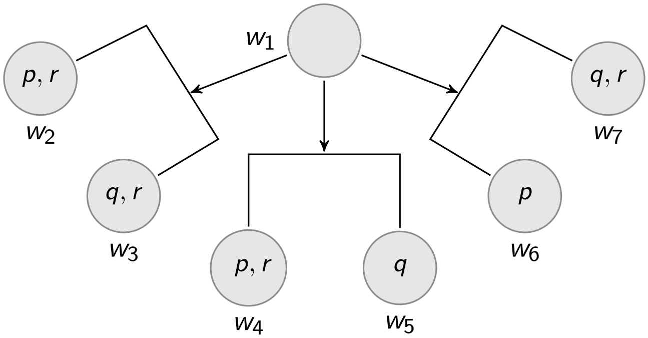
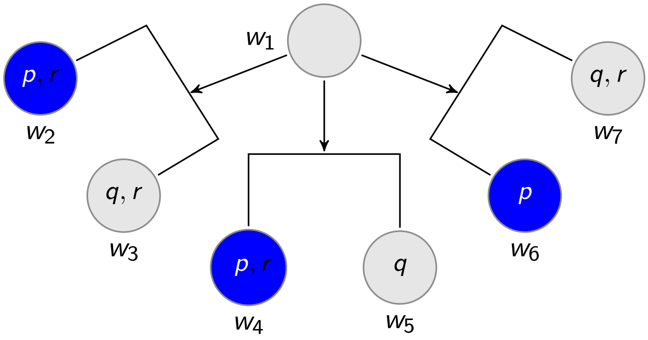
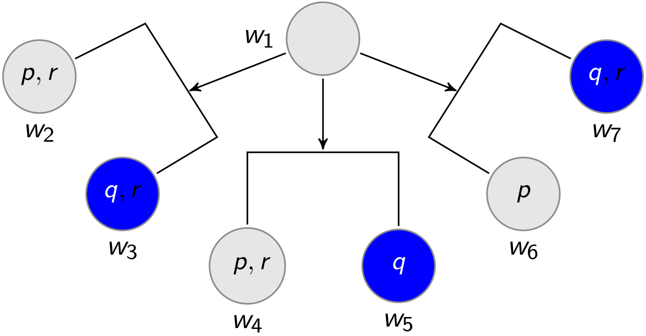
- $\mathcal{M}^n, w \models \Box\varphi$ iff for all $(w_1, \ldots, w_{n-1}) \in W^{n-1}$, if $(w, w_1, \ldots, w_n) \in R$, then there exists i such that $1 \leq i \leq n$ and $\mathcal{M}^n, w_i \models \varphi$.
- $\mathcal{M}^n, w \models \Diamond\varphi$ iff there exists $(w_1, \ldots, w_n) \in W^{n-2}$ such that $(w, w_1, \ldots, w_n) \in R$, and for all i such that $1 \leq i \leq n$, we have $\mathcal{M}^n, w_i \models \varphi$.



w_1
p, r
w_2
q, r
w_3
p, r
w_4
q
w_5
p
w_6
q, r
w_7



- $\mathcal{M}^3, w_1 \models \Box p$ (and $\mathcal{M}^3, w_1 \models \Box\neg p$)



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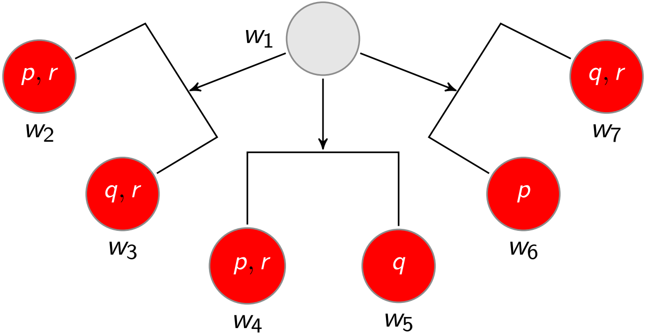

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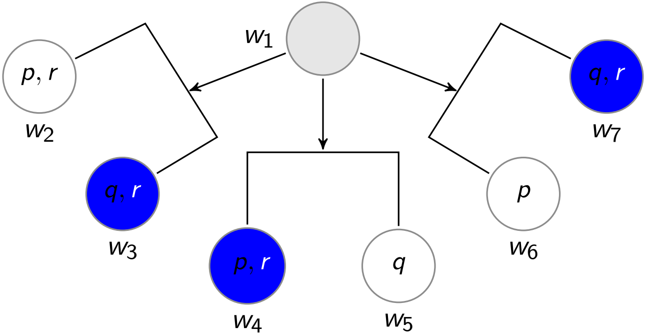
- $\mathcal{M}^3, w_1 \models \Box q$ (and $\mathcal{M}^3, w_1 \models \Box\neg q$)
- $\mathcal{M}^3, w_1 \not\models \Box(p \wedge q)$



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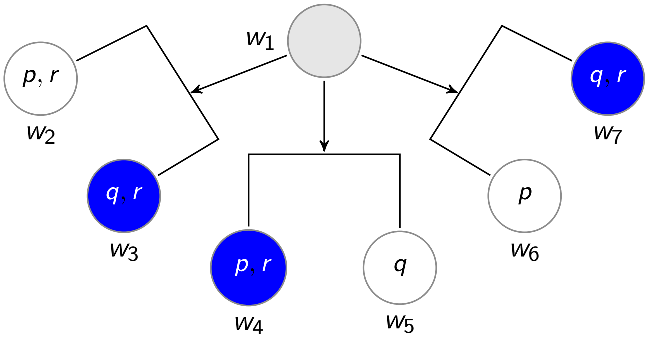
- $\mathcal{M}^3, w_1 \models \Box q$ (and $\mathcal{M}^3, w_1 \models \Box\neg q$)
- $\mathcal{M}^3, w_1 \not\models \Box(p \wedge q)$
- $\mathcal{M}^3, w_1 \models \Box r$



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- $\mathcal{M}^3, w_1 \not\models \Box(p \wedge q)$
- $\mathcal{M}^3, w_1 \models \Box r$
- $\mathcal{M}^3, w_1 \models \Box((p \wedge r) \vee (q \wedge r))$

$$(C^n) \qquad \bigwedge_{i=1}^{n} \Box\varphi_i \rightarrow \Box \bigvee_{1\leq k,l\leq n,k\neq l} (\varphi_k \wedge \varphi_l)$$

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Example:
$(\Box\varphi_1 \wedge \Box\varphi_2 \wedge \Box\varphi_3) \rightarrow \Box((\varphi_1 \wedge \varphi_2) \vee (\varphi_2 \wedge \varphi_3) \vee (\varphi_1 \wedge \varphi_3))$

Suppose that $\mathbf{L}(\mathfrak{C}^n) = \{\varphi \in \mathcal{L}(\mathsf{At}) \mid \text{for all } \mathcal{F}^n \in \mathfrak{C}^n, \mathcal{F}^n \models \varphi\}$.

$$\mathbf{EMN} = \bigcap_{n \geq 2} \mathbf{L}(\mathfrak{C}^n)$$

Theorem. The logic $\mathbf{EMNC}^n$ is sound and complete for the class $\mathfrak{C}^n$ of n-ary relational frames.

Proposition. Suppose that $\mathbb{M} = \langle W, N, V \rangle$ is finite monotonic neighborhood model such that for all $w \in W$, $N(w) \neq \emptyset$. Then, there is an n-ary relational model $\mathcal{M}^N = \langle W^N, R^N, V^N \rangle$ that is modally equivalent to $\mathbb{M}$.

Proposition. Suppose that $\mathcal{M}^n = \langle W, R, V \rangle$ is a finite n-ary relational model. Then, there is a finite monotonic neighborhood model $\mathbb{M}^R = \langle W^R, N^R, V^R \rangle$ that is modally equivalent to $\mathcal{M}^n$.

Multi-Relational Semantics/Non-Normal Modal Logics

A multi-relational Kripke model is a triple $\mathbb{M} = \langle W, \mathcal{R}, V \rangle$ where $\mathcal{R} \subseteq \wp(W \times W)$.

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Are multi-relational semantics *equivalent* to neighborhood semantics?

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A world is called impossible if nothing is necessary and everything is possible.

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A multi-relational model with impossible worlds is a quadruple $\mathbb{M} = \langle W, Q, \mathcal{R}, V \rangle$.

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A multi-relational model with impossible worlds is a quadruple $\mathbb{M} = \langle W, Q, \mathcal{R}, V\rangle$.

$\mathbb{M}, w \models \Box\varphi$ iff $w \notin Q$ and $\exists R \in \mathcal{R}$ such that $\forall v \in W$, if wRv then $\mathbb{M}, v \models \varphi$.

Multi-Relational Semantics/Non-Normal Modal Logics

M. Fitting. *Proof Methods for Modal and Intuitionistic Logics*. Synthese Library, 1983.

L. Goble. *Multiplex semantics for Deontic Logic*. Nordic Journal of Philosophical Logic, 5(2), pgs. 113-134, 2000.

G. Governatori and A. Rotolo. *On the axiomatization of Elgesem's logic of agency and ability*. Journal of Philosophical Logic, 34(4), pgs. 403 - 431, 2005.

Let $Th_{\mathcal{L}}(\mathcal{M}, w) = \{\varphi \in \mathcal{L} \mid \mathcal{M}, w \models \varphi\}$

Suppose that M and M$'$ are two classes of models for $\mathcal{L}$. Say that $\mathcal{M}, w$ is $\mathcal{L}$-equivalent to $\mathcal{M}', w'$, denoted $\mathcal{M}, w \equiv_{\mathcal{L}} \mathcal{M}', w'$, provided $Th_{\mathcal{L}}(\mathcal{M}, w) = Th_{\mathcal{L}}(\mathcal{M}', w')$.

A class of models M is $\mathcal{L}$-equivalent to a class of models M$'$ provided for each pointed model $\mathcal{M}, w$ from M, there exists a pointed model $\mathcal{M}', w'$ from M$'$ such that $\mathcal{M}, w \equiv_{\mathcal{L}} \mathcal{M}', w'$, and *vice versa*.

- The class $\mathsf{K} = \{\mathcal{M} \mid \mathcal{M}$ is a relational model $\}$ is modally equivalent to the class
  $\mathsf{M}_{aug} = \{\mathbb{M} \mid \mathbb{M}$ is an augmented neighborhood model$\}$

- The class $\mathsf{K}^n = \{\mathcal{M}^n \mid \mathcal{M}$ is an n-ary relational model $\}$ is modally equivalent to the class
  $\mathsf{M}_{reg} = \{\mathbb{M} \mid \mathbb{M}$ is a consistent regular neighborhood model$\}$

- The class $\mathsf{T} = \{\mathcal{M}^T \mid \mathcal{M}$ is a topological model $\}$ is modally equivalent to the class
  $\mathsf{M}_{\mathsf{S4}} = \{\mathbb{M} \mid \mathbb{M}$ is an **S4** neighborhood model$\}$

Core Theory

✓ Neighborhood Semantics in the Broader Logical Landscape
- Bisimulation
- Completeness, Decidability, Complexity
- Incompleteness
- Relation with Relational Semantics
- Model Theory

Expressive Power and Invariance

M. Pauly. *Bisimulation for Non-normal Modal Logic*. Manuscript, 1999.

H. Hansen. *Monotonic Modal Logic*. ILLC, Masters Thesis, 2003.

Monotonic Bisimulation

Suppose that $\mathfrak{M} = \langle W, N, V\rangle$ and $\mathfrak{M}' = \langle W', N', V'\rangle$ are two monotonic neighborhood models. A relation $Z \subseteq W \times W'$ is a **monotonic bisimulation** provided that, whenever wZw':

Atomic harmony: for each $p \in \text{At}$, $w \in V(p)$ iff $w' \in V'(p)$.

Zig: If $w\ N\ X$ then there is an $X' \subseteq W'$ such that $w'\ N'\ X'$ and $\forall x' \in X'$, $\exists x \in X$ such that $x\ Z\ x'$.

Zag: If $w'\ N'\ X'$ then there is an $X \subseteq W$ such that $w\ N\ X$ and $\forall x \in X$, $\exists x' \in X'$ such that $x\ Z\ x'$.

Write $\mathfrak{M}, w \underline{\leftrightarrow} \mathfrak{M}', w'$ when there is a monotonic bisimulation $Z \subseteq dom(\mathcal{M}) \times dom(\mathcal{M}')$ such that $w\ Z\ w'$.

Proposition. If $\mathcal{M}$ is a monotonic model, $\mathcal{M}, w \underline{\leftrightarrow} \mathcal{M}', w'$ implies $\mathcal{M}, w \equiv_{\mathcal{L}} \mathcal{M}', w'$.

Locally Core-Finite Models

Suppose that $\mathcal{F}$ is a monotonic collection of subsets of W. The **non-monotonic core**, denoted $\mathcal{F}^{nc}$, is a subset of $\mathcal{F}$ defined as follows:

$$\mathcal{F}^{nc} = \{X \mid X \in \mathcal{F} \text{ and for all } X' \subseteq W, \text{ if } X' \subseteq X, \text{ then } X' \not\in \mathcal{F}\}.$$

A monotonic collection of sets $\mathcal{F}$ is **core-complete** provided for all $X \in \mathcal{F}$, there exists a $Y \in \mathcal{F}^{nc}$ such that $Y \subseteq X$.

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A monotonic collection of sets $\mathcal{F}$ is **core-complete** provided for all $X \in \mathcal{F}$, there exists a $Y \in \mathcal{F}^{nc}$ such that $Y \subseteq X$.

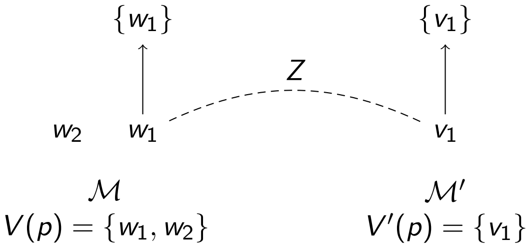
Question: Is every monotonic collection core-complete?

Locally Core-Finite Models

A neighborhood model $\mathcal{M} = \langle W, N, V \rangle$ is **locally core-finite** provided that $\mathcal{M}$ is core-complete and for each $w \in W$, $N^{nc}(w)$ is finite, and for all $X \in N^{nc}(w)$, X is finite.

Proposition. Suppose that $\mathcal{M} = \langle W, N, V \rangle$ and $\mathcal{M}' = \langle W', N', V' \rangle$ are monotonic, locally core-finite models. Then, for all $w \in W$, $w' \in W'$, $\mathcal{M}, w \equiv_{\mathcal{L}} \mathcal{M}', w'$ iff $\mathcal{M}, w \,\underline{\leftrightarrow}\, \mathcal{M}', w'$.

Do monotonic bisimulations work when we drop monotonicity? No!


{w1}
{v1}
Z
w2
w1
v1
M
V(p) = {w1, w2}
M′
V′(p) = {v1}

Bounded Morphisms

If $\mathcal{M}_1 = \langle W_1, N_1, V_1 \rangle$ and $\mathcal{M}_2 = \langle W_2, N_2, V_2 \rangle$ are two neighborhood models, and $f : W_1 \to W_2$ is a function, then f is a **(frame) bounded morphism** if

$$\text{for all } X \subseteq W_2, \text{ we have } f^{-1}[X] \in N_1(w) \text{ iff } X \in N_2(f(w));$$

and for all $p \in \mathsf{At}$, and all $w \in W_1$: $w \in V_1(p)$ iff $f(s) \in V_2(p)$.

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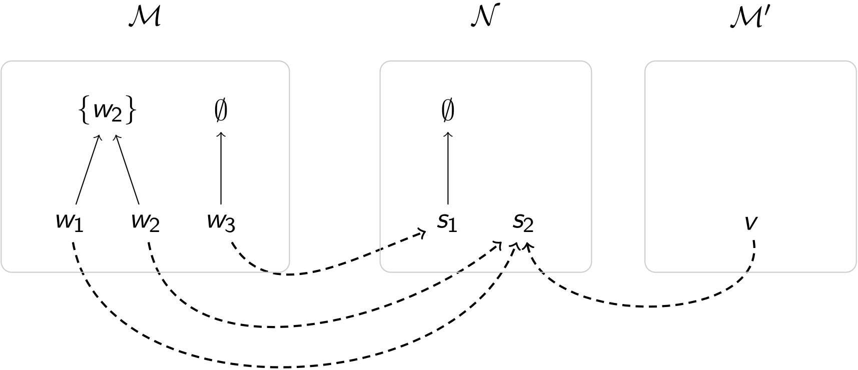
Lemma Let $\mathcal{M}_1 = \langle W_1, N_1, V_1 \rangle$ and $\mathcal{M}_2 = \langle W_2, N_2, V_2 \rangle$ be two neighborhood models and $f : \mathcal{M}_1 \to \mathcal{M}_2$ a bounded morphism. For each modal formula $\varphi \in \mathcal{L}$ and state $w \in W_1$, $\mathcal{M}_1, w \models \varphi$ iff $\mathcal{M}_2, f(w) \models \varphi$.

Behavioral Equivalence

Definition

Two points w_1 from $\mathfrak{M}_1$ and w_2 from $\mathfrak{M}_2$ are **behaviorally equivalent** provided there is a neighborhood frame $\mathfrak{F}$ and bounded morphisms $f : \mathfrak{F}_1 \to \mathfrak{F}$ and $g : \mathfrak{F}_2 \to \mathfrak{F}$ such that $f(w_1) = g(w_2)$.



$\mathcal{M}$
$\mathcal{N}$
$\mathcal{M}'$
$\{w_2\}$
$\emptyset$
$\emptyset$
w_1
w_2
w_3
s_1
s_2
v

Proposition. Suppose that $\mathcal{M} = \langle W, N, V\rangle$ and $\mathcal{M}' = \langle W', N', V'\rangle$ are two neighborhood models. If states $w \in W$ and $w' \in W'$ are behaviorally equivalent, then for all $\varphi \in \mathcal{L}$, $\mathcal{M}, w \models \varphi$ iff $\mathcal{M}', w' \models \varphi$.

Disjoint Union

Let $\mathcal{M}_1 = \langle W_1, N_1, V_1 \rangle$ and $\mathcal{M}_2 = \langle W_2, N_2, V_2 \rangle$ be two neighborhood models. The **disjoint union of $\mathcal{M}_1$ and $\mathcal{M}_2$** is the neighborhood model $\mathcal{M}_1 + \mathcal{M}_2 = \langle W_1 + W_2, N, V \rangle$ where for all $p \in \mathsf{At}$, $V(p) = V_1(p) \cup V_2(p)$; and for $i = 1, 2$,

for all $X \subseteq W_1 + W_2$, and $w \in W_i$, $X \in N(w)$ iff $X \cap W_i \in N_i(w)$.

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for all $X \subseteq W_1 + W_2$, and $w \in W_i$, $X \in N(w)$ iff $X \cap W_i \in N_i(w)$.

Proposition. For all $\varphi \in \mathcal{L}$, for $i = 1, 2$, if $w \in W_i$, then $\mathcal{M}_1 + \mathcal{M}_2, w \models \varphi$ iff $\mathcal{M}_i, w \models \varphi$.

Fact. The universal modality is not definable in the basic modal language.

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