Neighborhood Semantics for Modal Logic
Lecture 2

Eric Pacuit

University of Maryland, College Park
pacuit.org
epacuit@umd.edu

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Text

Neighborhood semantics for modal logic (Draft)

Ch 1: Introduction and Motivation

Ch 2: Core Theory: Expressivity, Completeness, Decidability, Complexity, Correspondence Theory

Ch 3: Richer Languages: Fixed-point operators, First-order extensions, Dynamic operators

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Ch 1: Introduction and Motivation

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Ch 3: Richer Languages: Fixed-point operators, First-order extensions, Dynamic operators

Current research papers non-normal modal logics and neighborhood structures.

Schedule

Lecture 1: June 1st, 14h00-16h30

Lecture 2: June 2nd 12h30-14h30

Lecture 3: June 7th, 14h00-16h30

Lecture 4: June 8th, 11h00-13h00

Lecture 5: June 8th, 14h00-16h30

Lecture 7: June 9th, 12h30-14h30

Lecture 8: June 13th,12h30-15h00

Lecture 9: June 14th, 10h00-13h00

Lecture 10 Presentations (solutions to problems etc.): June 15th, 10h00-13h00

Location: F1.15 ILLC

1. Non-normal modal logics
2. Neighborhood semantics for modal logic

Non-normal modal logics

(M)	$\Box(\varphi \wedge \psi) \rightarrow \Box\varphi \wedge \Box\psi$
(C)	$\Box\varphi \wedge \Box\psi \rightarrow \Box(\varphi \wedge \psi)$
(N)	$\Box\top$
(K)	$\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$
(Dual)	$\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$
(Nec)	from $\vdash \varphi$ infer $\vdash \Box\varphi$
(Re)	from $\vdash \varphi \leftrightarrow \psi$ infer $\vdash \Box\varphi \leftrightarrow \Box\psi$

Non-normal modal logics

(M) ~~$\Box(\varphi \wedge \psi) \rightarrow \Box\varphi \wedge \Box\psi$~~

(C) ~~$\Box\varphi \wedge \Box\psi \rightarrow \Box(\varphi \wedge \psi)$~~

(N) ~~$\Box\top$~~

(K) ~~$\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$~~

(Dual) $\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$

(Nec) ~~from $\vdash \varphi$ infer $\vdash \Box\varphi$~~

(Re) from $\vdash \varphi \leftrightarrow \psi$ infer $\vdash \Box\varphi \leftrightarrow \Box\psi$

PC Propositional Calculus

E $\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$

M $\Box(\varphi \wedge \psi) \rightarrow (\Box\varphi \wedge \Box\psi)$

C $(\Box\varphi \wedge \Box\psi) \rightarrow \Box(\varphi \wedge \psi)$

N $\Box\top$

K $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$

RE $\dfrac{\varphi \leftrightarrow \psi}{\Box\varphi \leftrightarrow \Box\psi}$

Nec $\dfrac{\varphi}{\Box\varphi}$

MP $\dfrac{\varphi \quad \varphi \rightarrow \psi}{\psi}$

PC Propositional Calculus

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K $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$

RE $$\frac{\varphi \leftrightarrow \psi}{\Box\varphi \leftrightarrow \Box\psi}$$

Nec $$\frac{\varphi}{\Box\varphi}$$

MP $$\frac{\varphi \quad \varphi \rightarrow \psi}{\psi}$$

A modal logic **L** is classical if it contains all instances of *E* and is closed under *RE*.

PC Propositional Calculus

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E is the smallest classical modal logic.

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E is the smallest classical modal logic.

In **E**, *M* is equivalent to

$$(Mon)\ \frac{\varphi \rightarrow \psi}{\Box\varphi \rightarrow \Box\psi}$$

PC Propositional Calculus

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Mon $\dfrac{\varphi \rightarrow \psi}{\Box\varphi \rightarrow \Box\psi}$

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E is the smallest classical modal logic.

EM is the logic **E** + *Mon*

PC 6. Propositional Calculus

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E is the smallest classical modal logic.

EM is the logic **E** + *Mon*

EC is the logic **E** + *C*

PC Propositional Calculus

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EM is the logic **E** + *Mon*

EC is the logic **E** + *C*

EMC is the smallest regular modal logic

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EM is the logic **E** + *Mon*

EC is the logic **E** + *C*

EMC is the smallest regular modal logic

A logic is normal if it contains all instances of *E*, *C* and is closed under *Mon* and *Nec*

PC Propositional Calculus

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E is the smallest classical modal logic.

EM is the logic **E** + *Mon*

EC is the logic **E** + *C*

EMC is the smallest regular modal logic

K is the smallest normal modal logic

PC Propositional Calculus

E $\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$

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E is the smallest classical modal logic.

EM is the logic **E** + *Mon*

EC is the logic **E** + *C*

EMC is the smallest regular modal logic

K = **EMCN**

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EMC is the smallest regular modal logic

K $= PC(+E) + K + Nec + MP$

Neighborhood Frames

Let W be a non-empty set of states.

Any function $N : W \to \wp(\wp(W))$ is called a neighborhood function

A pair $\langle W, N \rangle$ is a called a neighborhood frame if W a non-empty set and N is a neighborhood function.

A neighborhood model based on $\mathfrak{F} = \langle W, N \rangle$ is a tuple $\langle W, N, V \rangle$ where $V : \text{At} \to \wp(W)$ is a valuation function.

Truth in a Model

- $\mathfrak{M}, w \models p$ iff $w \in V(p)$
- $\mathfrak{M}, w \models \neg\varphi$ iff $\mathfrak{M}, w \not\models \varphi$
- $\mathfrak{M}, w \models \varphi \wedge \psi$ iff $\mathfrak{M}, w \models \varphi$ and $\mathfrak{M}, w \models \psi$

Truth in a Model

- $\mathfrak{M}, w \models p$ iff $w \in V(p)$
- $\mathfrak{M}, w \models \neg\varphi$ iff $\mathfrak{M}, w \not\models \varphi$
- $\mathfrak{M}, w \models \varphi \wedge \psi$ iff $\mathfrak{M}, w \models \varphi$ and $\mathfrak{M}, w \models \psi$
- $\mathfrak{M}, w \models \Box\varphi$ iff $[\![\varphi]\!]_{\mathfrak{M}} \in N(w)$
- $\mathfrak{M}, w \models \Diamond\varphi$ iff $W - [\![\varphi]\!]_{\mathfrak{M}} \notin N(w)$

where $[\![\varphi]\!]_{\mathfrak{M}} = \{w \mid \mathfrak{M}, w \models \varphi\}$.

Let $N : W \to \wp\wp W$ be a neighborhood function and define $m_N : \wp W \to \wp W$:

$$\text{for } X \subseteq W,\ m_N(X) = \{w \mid X \in N(w)\}$$

1. $[\![p]\!]_{\mathfrak{M}} = V(p)$ for $p \in \text{At}$
2. $[\![\neg\varphi]\!]_{\mathfrak{M}} = W - [\![\varphi]\!]_{\mathfrak{M}}$
3. $[\![\varphi \wedge \psi]\!]_{\mathfrak{M}} = [\![\varphi]\!]_{\mathfrak{M}} \cap [\![\psi]\!]_{\mathfrak{M}}$
4. $[\![\Box\varphi]\!]_{\mathfrak{M}} = m_N([\![\varphi]\!]_{\mathfrak{M}})$
5. $[\![\Diamond\varphi]\!]_{\mathfrak{M}} = W - m_N(W - [\![\varphi]\!]_{\mathfrak{M}})$

Detailed Example

Suppose $W = \{w, s, v\}$ is the set of states and define a neighborhood model $\mathfrak{M} = \langle W, N, V \rangle$ as follows:

- $N(w) = \{\{s\}, \{v\}, \{w, v\}\}$
- $N(s) = \{\{w, v\}, \{w\}, \{w, s\}\}$
- $N(v) = \{\{s, v\}, \{w\}, \emptyset\}$

Further suppose that $V(p) = \{w, s\}$ and $V(q) = \{s, v\}$.

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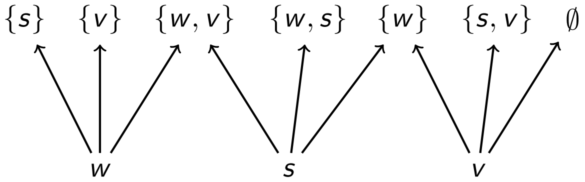

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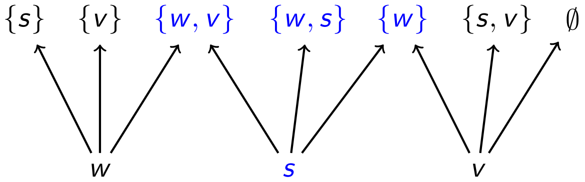

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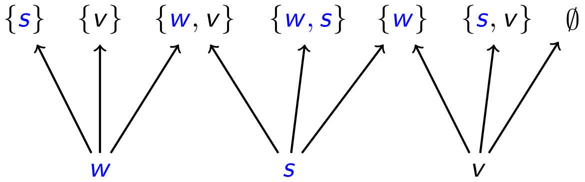
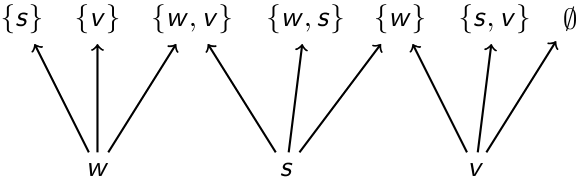


Detailed Example

$V(p) = \{w, s\}$ and $V(q) = \{s, v\}$



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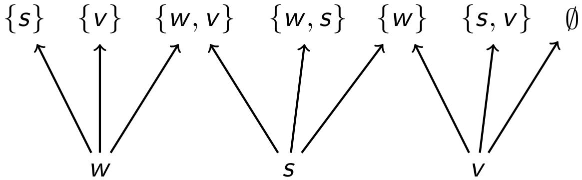
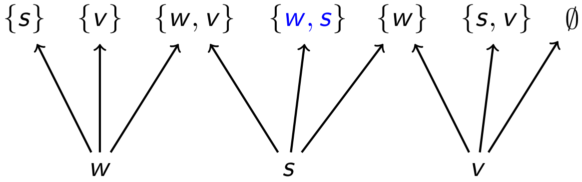


$$\mathfrak{M}, s \models \Box p$$

Detailed Example

$V(p) = \{w, s\}$ and $V(q) = \{s, v\}$



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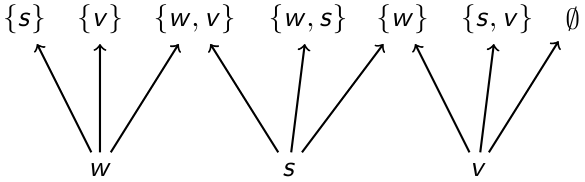
Detailed Example

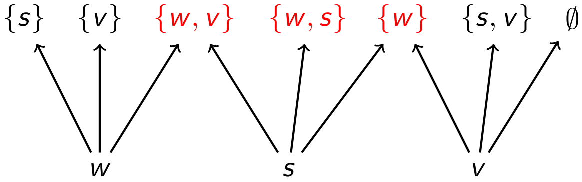
$V(p) = \{w, s\}$ and $V(q) = \{s, v\}$



$$\mathfrak{M}, s \models \Diamond p$$

Detailed Example

$V(p) = \{w, s\}$ and $V(q) = \{s, v\}$

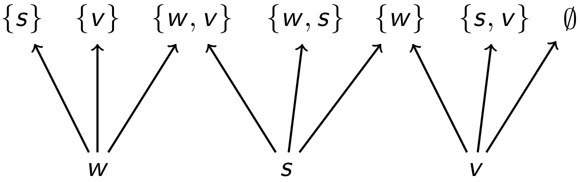


$$\mathfrak{M}, s \models \Diamond p$$

$$[\![\neg p]\!]_{\mathfrak{M}} = \{v\}$$

Detailed Example

$V(p) = \{w, s\}$ and $V(q) = \{s, v\}$



$\mathfrak{M}, w \models \Diamond\Box p$?

$\mathfrak{M}, w \models \Box\Box p$?

$\mathfrak{M}, v \models \Box\Diamond p$?

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Detailed Example

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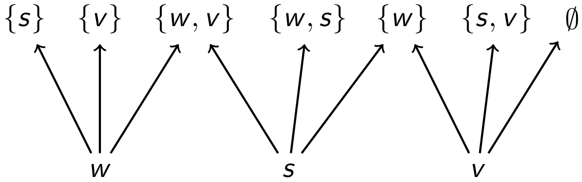

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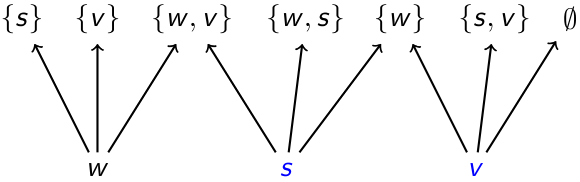

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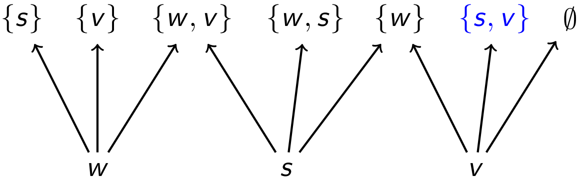

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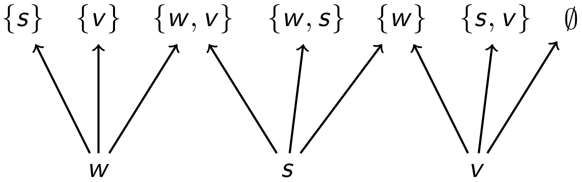

$\mathfrak{M}, w \not\models \Diamond\Box p$

$\mathfrak{M}, w \models \Box\Box p$

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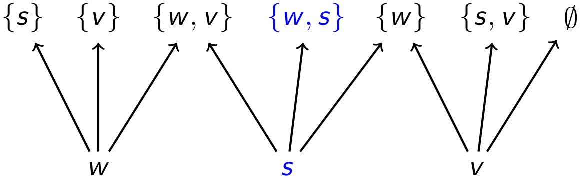
Detailed Example

$V(p) = \{w, s\}$ and $V(q) = \{s, v\}$



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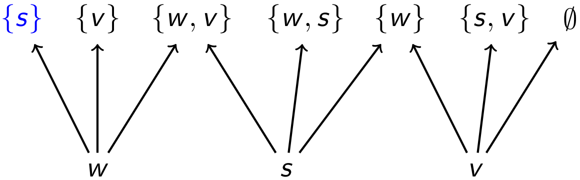

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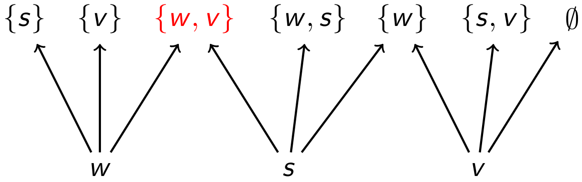

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$\mathfrak{M}, v \models \Diamond\Box p$

Other modal operators

- $\mathfrak{M}, w \models \langle\ \rangle\varphi$ iff $\exists X \in N(w)$ such that $\exists v \in X$, $\mathfrak{M}, v \models \varphi$
- $\mathfrak{M}, w \models [\ ]\varphi$ iff $\forall X \in N(w)$ such that $\forall v \in X$, $\mathfrak{M}, v \models \varphi$

- $\mathfrak{M}, w \models \langle\ ]\varphi$ iff $\exists X \in N(w)$ such that $\forall v \in X$, $\mathfrak{M}, v \models \varphi$
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Why non-normal modal logic?

Why neighborhood models?

Abilities

$Abl_i\varphi$: *i* has the ability to *see to it that* φ is true
(alternatively, *i* has the ability to bring about φ)

What are the core logical principles?

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2. $\neg Abl_i\top$

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What are the core logical principles?

1. $Abl_i\varphi \to \varphi$ (or $\varphi \to Abl_i\varphi$)
2. $\neg Abl_i\top$
3. $(Abl_i\varphi \wedge Abl_i\psi) \to Abl_i(\varphi \wedge \psi)$

Abilities

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3. $(Abl_i\varphi \wedge Abl_i\psi) \rightarrow Abl_i(\varphi \wedge \psi)$
4. $Abl_i(\varphi \vee \psi) \rightarrow (Abl_i\varphi \vee Abl_i\psi)$

Abilities

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4. $Abl_i(\varphi \vee \psi) \to (Abl_i\varphi \vee Abl_i\psi)$
5. $Abl_i(\varphi \wedge \psi) \to (Abl_i\varphi \wedge Abl_i\psi)$

Abilities

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5. $Abl_i(\varphi \wedge \psi) \rightarrow (Abl_i\varphi \wedge Abl_i\psi)$
6. $Abl_iAbl_j\varphi \rightarrow Abl_i\varphi$, $Abl_iAbl_i\varphi \rightarrow Abl_i\varphi$

Games: $(Abl_i\varphi \wedge Abl_i\psi) \not\to Abl_i(\varphi \wedge \psi)$

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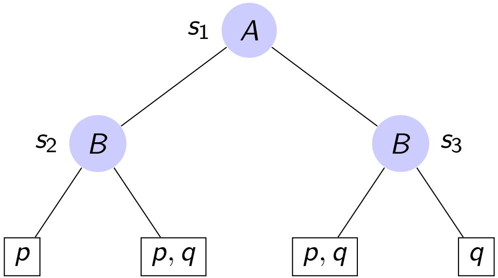

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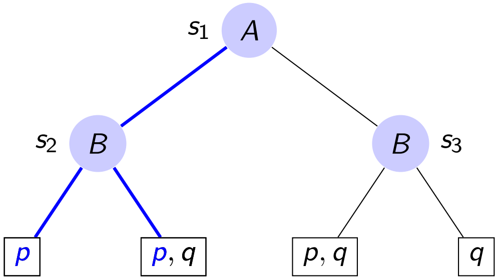

$s_1 \models Abl_A p$

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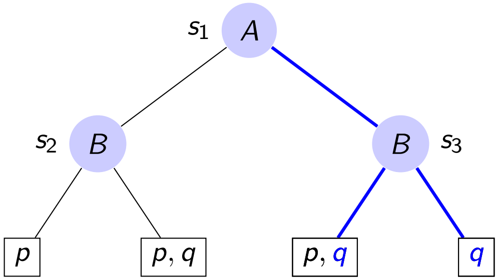

$$s_1 \models Abl_A p \wedge Abl_A q$$

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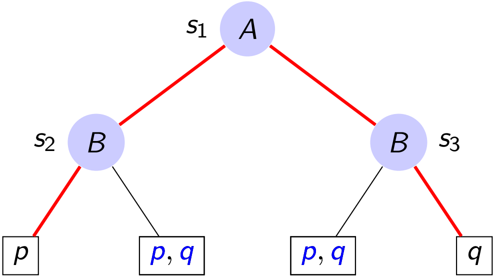

$$s_1 \models Abl_A p \wedge Abl_A q \wedge \neg Abl_A(p \wedge q)$$

Games: $(Abl_i\varphi \wedge Abl_i\psi) \not\to Abl_i(\varphi \wedge \psi)$

R. Parikh. *The Logic of Games and its Applications*. Annals of Discrete Mathematics (1985).

M. Pauly and R. Parikh. *Game Logic — An Overview*. Studia Logica (2003).

J. van Benthem. *Logic and Games*. Course notes (2007).

Question

$\Box_i \varphi$ means "player i has a strategy to win the game"

$\Diamond_i \varphi$ means "player i's opponent has a strategy to win the game"

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$\Box_i\varphi$ means "player i has a strategy to win the game"

$\Diamond_i\varphi$ means "player i's opponent has a strategy to win the game"

- Is $\neg\Diamond_i\neg\varphi \rightarrow \Box_i\varphi$ valid?
- Is $\Box_i\varphi \rightarrow \neg\Diamond_i\neg\varphi$ valid? Hint: the formula is equivalent to $\neg(\Box_i\varphi \wedge \Diamond_i\neg\varphi)$

$\varphi \not\to Abl_i\varphi$

Suppose an agent (call her Ann) is throwing a dart and she is not a very good dart player, but she just happens to throw a bull's eye.

$\varphi \not\to Abl_i\varphi$

Suppose an agent (call her Ann) is throwing a dart and she is not a very good dart player, but she just happens to throw a bull's eye.

Intuitively, we do not want to say that Ann has the *ability* to throw a bull's eye even though it happens to be true.

$Abl_i(\varphi \vee \psi) \not\to Abl_i\varphi \vee Abl_i\psi$

Continuing with this example, suppose that Ann has the ability to hit the dart board, but has no other control over the placement of the dart.

$Abl_i(\varphi \vee \psi) \not\rightarrow Abl_i\varphi \vee Abl_i\psi$

Continuing with this example, suppose that Ann has the ability to hit the dart board, but has no other control over the placement of the dart.

Now, when she throws the dart, as a matter of fact, it will either hit the top half of the board or the bottom half of the board.

$Abl_i(\varphi \vee \psi) \not\rightarrow Abl_i\varphi \vee Abl_i\psi$

Continuing with this example, suppose that Ann has the ability to hit the dart board, but has no other control over the placement of the dart.

Now, when she throws the dart, as a matter of fact, it will either hit the top half of the board or the bottom half of the board.

Since, Ann has the ability to hit the dart board, she has the ability to either hit the top half of the board or the bottom half of the board.

$Abl_i(\varphi \vee \psi) \not\rightarrow Abl_i\varphi \vee Abl_i\psi$

Continuing with this example, suppose that Ann has the ability to hit the dart board, but has no other control over the placement of the dart.

Now, when she throws the dart, as a matter of fact, it will either hit the top half of the board or the bottom half of the board.

Since, Ann has the ability to hit the dart board, she has the ability to either hit the top half of the board or the bottom half of the board.

However, intuitively, it seems true that Ann does not have the ability to hit the top half of the dart board, and also she does not have the ability to hit the bottom half of the dart board.

Abilities

$Abl_i\varphi$: agent i has the ability to bring about (see to it that) φ is true

What are core logical principles? Depends very much on the intended "application" and how actions are represented...

1. $Abl_i\varphi \to \varphi$ (or $\varphi \to Abl_i\varphi$)
2. $\neg Abl_i\top$
3. $(Abl_i\varphi \wedge Abl_i\psi) \to Abl_i(\varphi \wedge \psi)$
4. $Abl_i(\varphi \vee \psi) \to (Abl_i\varphi \vee Abl_i\psi)$
5. $Abl_i(\varphi \wedge \psi) \to (Abl_i\varphi \wedge Abl_i\psi)$
6. $Abl_iAbl_j\varphi \to Abl_i\varphi$, $Abl_iAbl_i\varphi \to Abl_i\varphi$

On the Logic of Ability

$$Abl_i\top$$

$$\varphi \rightarrow Abl_i\varphi$$

$$(Abl_i\varphi \wedge Abl_i\psi) \rightarrow Abl_i(\varphi \wedge \psi)$$

$$Abl_i(\varphi \vee \psi) \rightarrow (Abl_i\varphi \vee Abl_i\psi)$$

On the Logic of Ability

$$\neg Abl_i \top$$

$$\varphi \not\to Abl_i \varphi$$

$$(Abl_i \varphi \wedge Abl_i \psi) \not\to Abl_i(\varphi \wedge \psi)$$

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On the Logic of Ability

$\neg Abl_i \top$

$\Box\top$ is valid in the class of all frames, $\Diamond\top$ is valid on the class of serial frames

$\varphi \not\to Abl_i\varphi$

$(Abl_i\varphi \wedge Abl_i\psi) \not\to Abl_i(\varphi \wedge \psi)$

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$(\Box\varphi \wedge \Box\psi) \to \Box(\varphi \wedge \psi)$ is valid in the class of all frames

$Abl_i(\varphi \vee \psi) \not\to (Abl_i\varphi \vee Abl_i\psi)$

$\Diamond(\varphi \vee \psi) \to (\Diamond\varphi \vee \Diamond\psi)$ is valid in the class of all frames

Ability: Reproducibility vs. Reliability

"Abilities are inherently general; there are no genuine abilities which are abilities to do things only on one particular occasion"

(p. 135)

A. Kenny. *Will, Freedom and Power*. 1975.

Ability: Reproducibility vs. Reliability

"Abilities are inherently general; there are no genuine abilities which are abilities to do things only on one particular occasion"

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A. Kenny. *Will, Freedom and Power*. 1975.

"Even if opportunity only knocks once, I may be able to act on it, and may be culpable for doing so, or for failing to do so."

(p. 1)

M. Brown. *On the Logic of Ability*. Journal of Philosophical Logic, Vol. 17, pp. 1 - 26, 1988.

D. Elgesem. *The modal logic of agency*. Nordic Journal of Philosophical Logic 2(2), 1 - 46, 1997.

G. Governatori and A. Rotolo. *On the Axiomatisation of Elgesem's Logic of Agency and Ability*. Journal of Philosophical Logic, 34, pgs. 403 - 431 (2005).

A Minimal Logic of Abilities

$C\varphi$ means "the agent is capable of realizing φ"

$E\varphi$ means "the agent does bring about φ"

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$C\varphi$ means "the agent is capable of realizing φ"

$E\varphi$ means "the agent does bring about φ"

1. All propositional tautologies
2. $\neg C\top$
3. $E\varphi \wedge E\psi \rightarrow E(\varphi \wedge \psi)$
4. $E\varphi \rightarrow \varphi$
5. $E\varphi \rightarrow C\varphi$
6. Modus Ponens plus from $\varphi \leftrightarrow \psi$ infer $E\varphi \leftrightarrow E\psi$ and from $\varphi \leftrightarrow \psi$ infer $C\varphi \leftrightarrow C\psi$

Social Choice Theory

$\Box\alpha$ mean "*the group accepts* α."

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Note: the language is restricted so that $\Box\Box\alpha$ *is not a wff.*

Social Choice Theory

$\Box\alpha$ mean "*the group accepts* α."

Consensus: α is accepted provided *everyone* accepts α.

(E) $\Box\alpha \leftrightarrow \Box\beta$ provided $\alpha \leftrightarrow \beta$ is a tautology

(M) $\Box(\alpha \wedge \beta) \rightarrow (\Box\alpha \wedge \Box\beta)$

(C) $(\Box\alpha \wedge \Box\beta) \rightarrow (\Box\alpha \wedge \Box\beta)$

(N) $\Box\top$

(D) $\neg\Box\bot$

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(D) $\neg\Box\bot$

Theorem The above axioms axiomatize consensus (provided $n \geq 2^{|At|}$)).

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(M) $\Box(\alpha \wedge \beta) \rightarrow (\Box\alpha \wedge \Box\beta)$

(S) $\Box\alpha \rightarrow \neg\Box\neg\alpha$

(T) $([\geq]\varphi_1 \wedge \cdots \wedge [\geq]\varphi_k \wedge [\leq]\psi_1 \wedge \cdots \wedge [\leq]\psi_k) \rightarrow \bigwedge_{1\leq i\leq k}([=]\varphi_i \wedge [=]\psi_i)$ where $\forall v \in V_I$: $|\{i \mid v_(\varphi_i) = 1\}| = |\{i \mid v(\psi_i) = 1\}|$

Theorem The above axioms axiomatize majority rule.

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Majority: α is accepted if a *majority* of the agents accept α.

Why is $\Box\alpha \wedge \Box\beta \rightarrow \Box(\alpha \wedge \beta)$ invalid?

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Why is $\Box\alpha \wedge \Box\beta \to \Box(\alpha \wedge \beta)$ invalid?

	p	q	$p \wedge q$
i	1	1	1
j	1	0	0
k	0	1	0
Majority	1	1	0

Social Choice Theory

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M. Pauly. *Axiomatizing Collective Judgement Sets in a Minimal Logical Language*. 2006.

T. Daniëls. *Social Choice and Logic via Simple Games*. ILLC, Masters Thesis, 2007.

- ✓ Logical omniscience
- ✓ Logics of knowledge and beliefs
- ✓ Logic of high probability
- ✓ Logics of classical deduction
- ✓ Deontic logics
- ✓ Logics of ability
- ✓ Logics of group decision making
- ???

Why non-normal modal logic? ✓

Why neighborhood models?

✓ Subset spaces, neighborhood frames/models, reasoning about subset spaces
- Logic of knowledge, evidence and belief
- Coalitional logic
- Interesting mathematical structures: Ultrafilters, topologies, hypergraphs

A (Dynamic) Logic of Knowledge, Evidence and Belief

J. van Benthem and EP. *Dynamic Logics of Evidence-Based Beliefs*. Studia Logica, 99, pp. 61 - 92, 2011.

J. van Benthem, D. Fernández-Duque and EP. *Evidence Logic: A New Look at Neighborhood Structures*. Proceedings of Advances in Modal Logic, King's College Publications, 2012.

J. van Benthem, D. Fernández-Duque and EP. *Evidence and Plausibility in Neighborhood Structures*. Annals of Pure and Applied Logic, 2013.

Setting the Stage: Evidence

- Dempster-Shafer Theory of Evidence

G. Shafer. *A Mathematical Theory of Evidence*. Princeton University Press, 1976.

- Bayesian Confirmation Theory (eg., E confirms H iff $p(H \mid E) > p(H)$)

B. Fitelson. *The Plurality of Bayesian Measures of Confirmation and the Problem of Measure Sensitivity*. Philosophy of Science 66, 1999.

Setting the Stage: Evidence

- Artemov/Fitting's Justification Logic ($t:\varphi$: "t is a *justification/proof* for φ")

S. Artemov and M. Fitting. *Justification logic*. The Stanford Encyclopedia of Philosophy, 2012.

- Moss and Parikh's "topologic" ($x, U \models \varphi$: "φ is true at the state x given that the current *evidence/ "measurement"* gathered is U")

L. Moss and R. Parikh. *Topological reasoning and the logic of knowledge*. Proceedings of TARK IV, Morgan Kaufmann, 1992.

Setting the Stage: Reasons

- Kratzer Semantics (modal base), believing for a *reason* (deriving an ordering on worlds from an ordering over propositions)

A. Kratzer. *What* must *and* can *must and can mean*. Linguistics and Philosophy 1 (1977) 337355.

C. List and F. Dietrich. *Reasons for (prior) belief in bayesian epistemology*. 2012.

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C. List and F. Dietrich. *Reasons for (prior) belief in bayesian epistemology*. 2012.

- Reason management (Default logic with priorities)

J. Horty. *Reasons as Defaults*. 2012.

Modeling Evidence: Some Distinctions

Barest view: the evidence is encoded as the current range of worlds the agent considers possible

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Barest view: the evidence is encoded as the current range of worlds the agent considers possible

Ignores how we arrived at this epistemic state

Richest view: complete syntactic details of what we have learned so far (including the sources of each piece of evidence)

In between: family of subsets representing evidence from received from various (possible unreliable) sources

Evidence Models: Basic Assumptions

Let W be a set of possible worlds or states one of which represents the "actual" situation.

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2. The evidence gathered from different sources (or even the same source) may be jointly inconsistent. And so, the intersection of all the gathered evidence may be empty.
3. Despite the fact that sources may not be reliable or jointly inconsistent, they are all the agent has for forming beliefs.

Evidential States

An **evidential state** is a collection of subsets of W.

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Assumptions:

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Assumptions:

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- The whole universe W is an evidence set (agents know their 'space').

In addition, much of the literature would suggest a 'monotonicity' assumption:

> *If the agent has evidence X and $X \subseteq Y$ then the agent has evidence Y.*

Example: $W = \{w, v\}$ where p is true only at w

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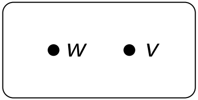
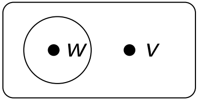
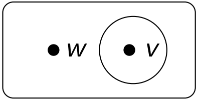


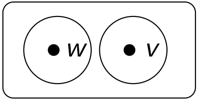
There is no evidence for or against p.



There is evidence that supports p.



There is evidence that rejects p.



There is evidence that supports p and also evidence that rejects p.

Evidence Models

Evidence model: $\mathcal{M} = \langle W, E, V \rangle$

- W is a non-empty set of worlds,
- $V : \mathsf{At} \to \wp(W)$ is a valuation function, and
- $E : W \to \wp(\wp(W))$ is an evidence relation

  $X \in E(w)$: "the agent accepts X as evidence at state w".

Uniform evidence model (E is a constant function):
$\langle W, \mathcal{E}, V \rangle, w$ where $\mathcal{E}$ is the fixed family of subsets of W related to each state by E.

Assumptions

(Cons) For each state w, $\emptyset \notin E(w)$.

(Triv) For each state w, $W \in E(w)$.

The Basic Language $\mathcal{L}$ of Evidence and Belief

$$p \mid \neg\varphi \mid \varphi \wedge \psi \mid \langle\,]\varphi \mid [B]\varphi \mid [A]\varphi$$

- $\langle\,]\varphi$ says that "the agent has evidence that φ is true" (i.e., "the agent has evidence for φ")
- $[B]\varphi$ says that "the agents believes that φ is true" (based on her evidence)
- $[A]\varphi$ says that "φ is true in all states" (which we interpret as the agent's *knowledge*)

Truth

- $\mathcal{M}, w \models p$ iff $w \in V(p) \qquad (p \in \mathsf{At})$
- $\mathcal{M}, w \models \neg\varphi$ iff $\mathcal{M}, w \not\models \varphi$
- $\mathcal{M}, w \models \varphi \wedge \psi$ iff $\mathcal{M}, w \models \varphi$ and $\mathcal{M}, w \models \psi$

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- $\mathcal{M}, w \models p$ iff $w \in V(p)$ $\qquad (p \in \text{At})$
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- $\mathcal{M}, w \models \langle\ ]\varphi$ iff there exists X such that $X \in E(w)$ and for all $v \in X$, $\mathcal{M}, v \models \varphi$

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- $\mathcal{M}, w \models \langle\ ]\varphi$ iff there exists X such that $X \in E(w)$ and for all $v \in X$, $\mathcal{M}, v \models \varphi$
- $\mathcal{M}, w \models [A]\varphi$ iff for all $v \in W$, $\mathcal{M}, v \models \varphi$

"Having evidence for φ" vs. "Accepting φ as evidence"

We do not assume that the evidence sets are closed under supersets, though our semantic definition implies that the set of propositions that the agent has *evidence for* is closed under weakening.

So, an agent can have *evidence for* X without *accepting* the set X as evidence.

Defining Beliefs

w-**scenario**: A maximal family of evidence sets $\mathcal{X} \subseteq E(w)$ that has the **finite intersection property** (f.i.p.: for each finite subfamily $\{X_1, \ldots, X_n\} \subseteq \mathcal{X}$, $\bigcap_{1 \leq i \leq n} X_i \neq \emptyset$).

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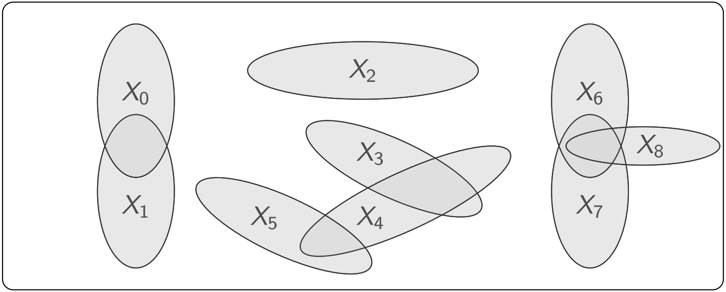
An agent believes φ at w if each w-scenario implies that φ is true (i.e., φ is true at each point in the intersection of each w-scenario).

Defining Beliefs



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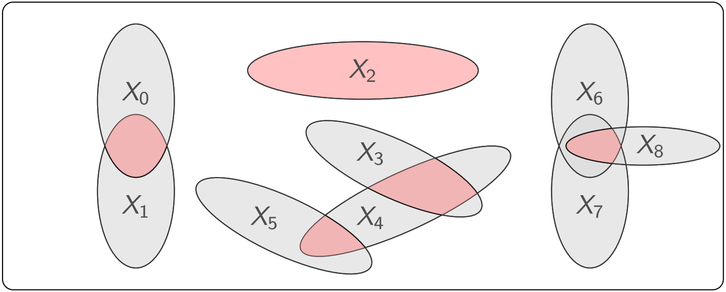

Our definition of belief is very conservative, many other definitions are possible (there exists a w-scenario, "most" of the w-scenarios,...)

Truth

- $\mathcal{M}, w \models p$ iff $w \in V(p)$ $\qquad (p \in \text{At})$
- $\mathcal{M}, w \models \neg\varphi$ iff $\mathcal{M}, w \not\models \varphi$
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- $\mathcal{M}, w \models \langle\ ]\varphi$ iff there exists X such that wEX and for all $v \in X$, $\mathcal{M}, v \models \varphi$
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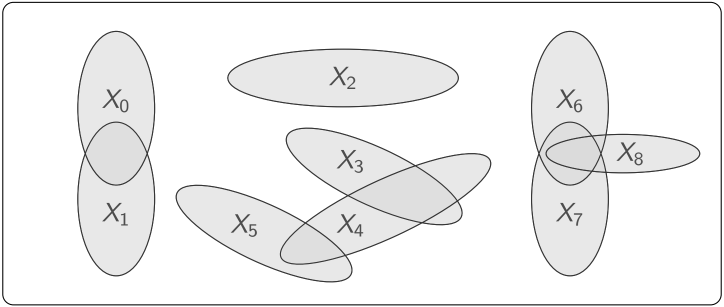
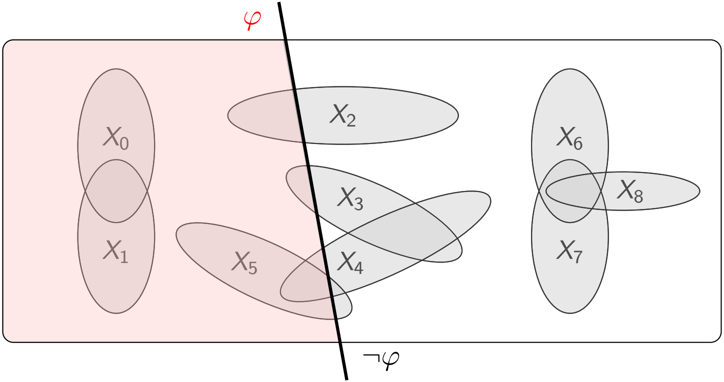
- $\mathcal{M}, w \models [A]\varphi$ iff for all $v \in W$, $\mathcal{M}, v \models \varphi$
- $\mathcal{M}, w \models [B]\varphi$ for all w-scenarios $\mathcal{X} \subseteq E(w)$, for all $v \in \bigcap \mathcal{X}$, $\mathcal{M}, v \models \varphi$

Notation for the truth set: $[\![\varphi]\!]_{\mathcal{M}} = \{w \mid \mathcal{M}, w \models \varphi\}$

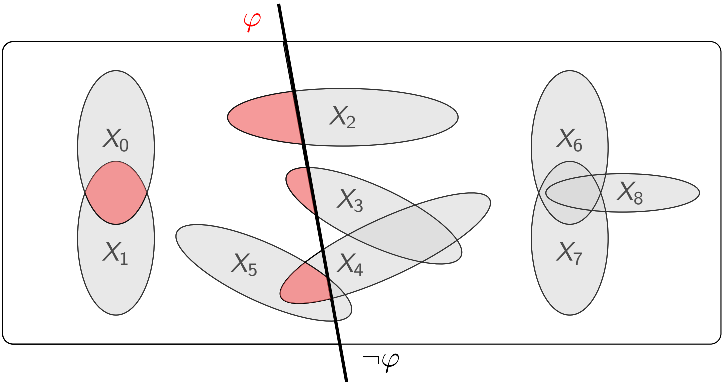
Conditional Beliefs on Evidence Models



Conditional Beliefs on Evidence Models



Conditional Beliefs on Evidence Models



Conditional Beliefs on Evidence Models

$B^{\varphi}\psi$: "the agent believes ψ conditional on φ."

Main idea: Ignore the evidence that is inconsistent with φ.

Relativized w**-scenario**: Suppose that $X \subseteq W$. Given a collection $\mathcal{X} \subseteq \wp(W)$, let $\mathcal{X}^X = \{Y \cap X \mid Y \in \mathcal{X}\}$. We say that a collection $\mathcal{X}$ of subsets of W has the **finite intersection property relative to** X (X-f.i.p.) if, $\mathcal{X}^X$ as the f.i.p. and is maximal if $\mathcal{X}^X$ is.

- $\mathcal{M}, w \models B^{\varphi}\psi$ iff for each maximal φ-f.i.p. $\mathcal{X} \subseteq E(w)$, for each $v \in \bigcap \mathcal{X}^{\varphi}$, $\mathcal{M}, v \models \psi$

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$B\psi \to B^{\varphi}\psi$ is not valid.

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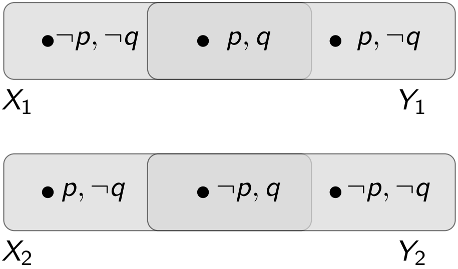

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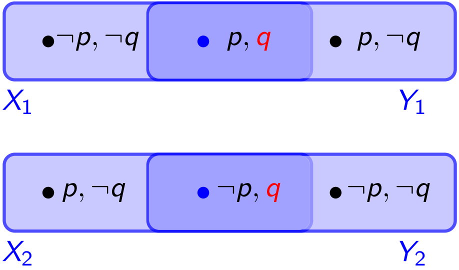

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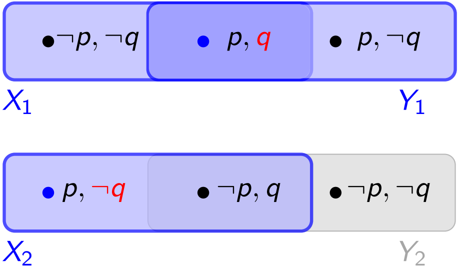

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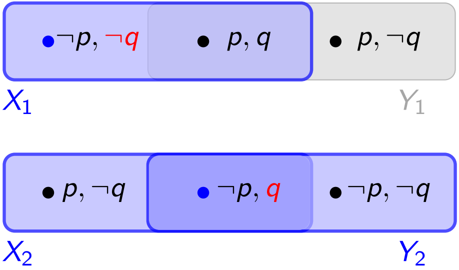

- ✓ $\mathcal{M}, w \models Bq$
- ✓ $\mathcal{M}, w \not\models B^{p}q$
- $\mathcal{M}, w \not\models B^{\neg p}q$

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$\langle\ ]^{\varphi}\psi$ is not equivalent to $\langle\ ](\varphi \rightarrow \psi)$: if there is no evidence consistent with φ, then $\langle\ ]^{\varphi}\psi$ is false.

Truth

- $\mathcal{M}, w \models p$ iff $w \in V(p) \qquad (p \in \mathsf{At})$
- $\mathcal{M}, w \models \neg\varphi$ iff $\mathcal{M}, w \not\models \varphi$
- $\mathcal{M}, w \models \varphi \wedge \psi$ iff $\mathcal{M}, w \models \varphi$ and $\mathcal{M}, w \models \psi$
- $\mathcal{M}, w \models \langle\ ]\varphi$ iff there exists X such that wEX and for all $v \in X$, $\mathcal{M}, v \models \varphi$
- $\mathcal{M}, w \models \langle\ ]^{\varphi}\psi$ iff there exists an evidence set $X \in E(w)$ consistent with φ such that for all $v \in X \cap [\![\varphi]\!]_{\mathcal{M}}$, $\mathcal{M}, v \models \psi$.
- $\mathcal{M}, w \models [A]\varphi$ iff for all $v \in W$, $\mathcal{M}, v \models \varphi$

Truth

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- $\mathcal{M}, w \models [A]\varphi$ iff for all $v \in W$, $\mathcal{M}, v \models \varphi$
- $\mathcal{M}, w \models [B]\varphi$ for all w-scenarios $\mathcal{X} \subseteq E(w)$, for all $v \in \bigcap \mathcal{X}$, $\mathcal{M}, v \models \varphi$
- $\mathcal{M}, w \models B^{\varphi}\psi$ iff for each maximal φ-f.i.p. $\mathcal{X} \subseteq E(w)$, for each $v \in \bigcap \mathcal{X}^{\varphi}$, $\mathcal{M}, v \models \psi$

Flat Evidence Models

An evidence model $\mathcal{M}$ is **flat** if every scenario on $\mathcal{M}$ has non-empty intersection.

Proposition. The formula $\langle\ ]\varphi \rightarrow \langle B\rangle\varphi$ is valid on the class of flat evidence models, but not on the class of all evidence models.

1. Prove that $\langle\ ]\varphi \wedge [A]\psi \leftrightarrow \langle\ ](\varphi \wedge [A]\psi)$ is valid on all evidence models.
2. Prove that $[B]\varphi \to [A][B]\varphi$ is valid on all uniform evidence models.
3. Show that $\langle\ ]\varphi \to \langle\ ]\langle\ ]\varphi$ is only valid on uniform evidence models.

- ✓ Subset spaces, neighborhood frames/models, reasoning about subset spaces
- ✓ Logic of knowledge, evidence and belief
- Coalitional logic
- Interesting mathematical structures: Ultrafilters, topologies, hypergraphs

Hypergraphs

directed graph: (W, E) where W is a non-empty set, elements of which are called **nodes** or **vertices**, and $E \subseteq W \times W$, elements of which are called **edges**

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hypergraph: $(W, \mathcal{E})$ where $\mathcal{E} \subseteq \wp(W)$ and $\emptyset \notin \mathcal{E}$.

A. Bretto. *Hypergraph Theory: An Introduction*. Springer, 2013.

A. Taylor W. S. Zwicker. *Simple Games: Desirability Relations, Trading, Pseudoweightings*. Princeton University Press, 1999.

Simple Games

Suppose that I is a finite set of voters.

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A simple game on I is a monotonic subset space $(I, \mathcal{W})$: $I \neq \emptyset$, $\mathcal{W} \subseteq \wp(I)$ such that for all $U, V \subseteq I$, if $U \in \mathcal{W}$ and $U \subseteq V$, then $V \in \mathcal{W}$.

Elements $U \in \mathcal{W}$ are called **winning coalitions**.

The intended interpretation is that the set U of voters is a **winning coalition** iff an option is selected by the group (e.g., the bill or amendment passes, or the candidate is elected) when the voters in U are the ones who voted for it.

Simple Games: Example

$I = \{a, b, c, d, e\}$ and consider the following winning coalitions:

$$\mathcal{W} = \{\{d, e\}, \{a, b, c, e\}, \{a, b, d\}, \{b, c, d\},$$
$$\{a, c, d\}, \{a, b, c, d\}, \{a, b, c, d, e\}\}$$

weighted simple game: $(I, \mathcal{W})$ is weighted if there is a function $weight : I \to \mathbb{R}$ and quota $q \in \mathbb{R}$, such that for all $U \subseteq I$, $U \in \mathcal{W}$ iff $\sum_{u \in U} weight(u) \geq q$.

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$$\mathcal{W} = \{\{d, e\}, \{a, b, c, e\}, \{a, b, d\}, \{b, c, d\}, \\ \{a, c, d\}, \{a, b, c, d\}, \{a, b, c, d, e\}\}$$

The above simple game is generated by the weight function $\mathit{weight} : \{a, b, c, d, e\} \to \mathbb{R}$ where $\mathit{weight}(a) = \mathit{weight}(b) = \mathit{weight}(c) = 1$, $\mathit{weight}(d) = 3$ and $\mathit{weight}(e) = 2$ with the quota $q = 5$.

Voting Blocs

If voters a and b always vote the same way, then they form a **voting bloc**

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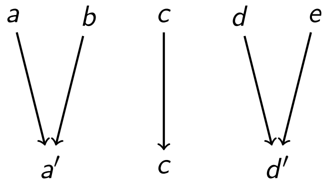
It is natural to identify voters that vote the same way.


a
b
c
d
e
a′
c
d′

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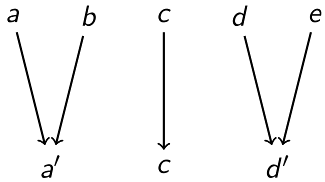

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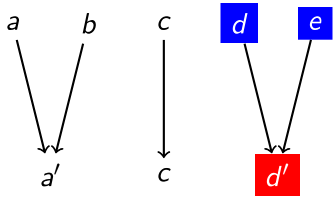

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Rudin-Keisler Ordering

Suppose that $\mathcal{G} = (I, \mathcal{W})$ is a simple game. The simple game $\mathcal{G}' = (I', \mathcal{W}')$ is a **RK-projection** of $(I, \mathcal{W})$, denoted $\mathcal{G}' \leq_{RK} \mathcal{G}$, if there is a surjective function $f : I \to I'$ such that for all $X \subseteq I'$, $X \in \mathcal{W}'$ iff $f^{-1}[X] \in \mathcal{W}$.

If $\mathcal{G}' \leq_{RK} \mathcal{G}$, then $\mathcal{G}'$ is said to be a RK-projection of $\mathcal{G}$.

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The Rudin-Keisler ordering is important because it preserves many properties of simple games.

Coalitional Logic

M. Pauly. *A Modal Logic for Coalitional Powers in Games*. Journal of Logic and Computation, 12:1, pp. 149 - 166, 2002.

M. Pauly. *Logic for Social Software*. PhD Thesis, Institute for Logic, Language and Computation, 2001.

Strategic Game Forms

$$\langle N, \{S_i\}_{i \in N}, O, o \rangle$$

- N is a finite set of players;
- for each $i \in N$, S_i is a non-empty set (elements of which are called actions or strategies);
- O is a non-empty set (elements of which are called **outcomes**); and
- $o : \Pi_{i \in N} S_i \to O$ is a function assigning an outcome

		Bob	
		t_1	t_2
Ann	s_1	o_1	o_2
	s_2	o_2	o_3
	s_3	o_4	o_1

α-Effectivity

$S = \Pi_{i \in N} S_i$ are called **strategy profiles**. Given a strategy profile $s \in S$, let s_i denote i's component and s_{-i} the profile of strategies from s for all players except i.

A strategy for a coalition C is a sequence of strategies for each player in C, i.e., $s_C \in \Pi_{i \in C} S_i$ (similarly for $s_{\overline{C}}$, where $\overline{C}$ is $N - C$).

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Suppose that $G = \langle N, \{S_i\}_{i \in N}, O, o \rangle$ be a strategic game form. An α-**effectivity function** is a map $E^\alpha_G : \wp(N) \to \wp(\wp(O))$ defined as follows: For all $C \subseteq N$, $X \in E^\alpha_G(C)$ iff there exists a strategy profile s_C such that for all $s_{\overline{C}} \in \Pi_{i \in N-C} S_i$, $o(s_C, s_{\overline{C}}) \in X$.

α-Effectivity vs. β-Effectivity

$\exists$ "something a player/a coalition *can* do" such that $\forall$ "actions of the other players/nature"...

α-Effectivity vs. β-Effectivity

$\exists$ "something a player/a coalition *can* do" such that $\forall$ "actions of the other players/nature"...

$\forall$ "(joint) actions of the other players", $\exists$ "something the agent/coalition can do"...

		Bob	
		t_1	t_2
Ann	s_1	o_1	o_2
	s_2	o_2	o_3
	s_3	o_4	o_1

Ann \ Bob	t_1	t_2
s_1	o_1	o_2
s_2	o_2	o_3
s_3	o_4	o_1

$$E^{\alpha}_{G_0}(\{A\}) = sup(\{\{o_1, o_2\}, \{o_2, o_3\}, \{o_1, o_4\}\})$$

Ann	Bob: t_1	t_2
s_1	o_1	o_2
s_2	o_2	o_3
s_3	o_4	o_1

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		Bob	
		t_1	t_2
Ann	s_1	o_1	o_2
	s_2	o_2	o_3
	s_3	o_4	o_1

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$$E^{\alpha}_{G_0}(\{A, B\}) = sup(\{\{o_1\}, \{o_2\}, \{o_3\}, \{o_4\}\}) = \wp(O) - \emptyset$$

Bob

		t_1	t_2
Ann	s_1	o_1	o_2
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	s_3	o_4	o_1

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$$E^{\alpha}_{G_0}(\emptyset) = \{\{o_1, o_2, o_3, o_4, o_5, o_6\}\}$$

Playable Effectivity Functions

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4. (*Outcome-monotonicity*) For all $X \subseteq X' \subseteq O$, and $C \subseteq N$, if $X \in E(C)$ then $X' \in E(C)$
5. (*Superadditivity*) For all subsets X_1, X_2 of O and sets of agents C_1, C_2, if $C_1 \cap C_2 = \emptyset$, $X_1 \in E(C_1)$ and $X_2 \in E(C_2)$, then $X_1 \cap X_2 \in E(C_1 \cup C_2)$

$E(\{i\}) = \{X \mid X \subseteq \mathbb{N} \text{ is infinite}\}$;
$E(\emptyset) = \{X \mid X \subseteq \mathbb{N} \text{ is cofinite (i.e., } \overline{X} \text{ is finite)}\}$;

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Claim. E satisfies Liveness, Safety, N-maximality, Outcome Monotonicity, Superadditivity, but is not the effectivity function of any game.

Core-Complete

Suppose that $(W, \mathcal{F})$ is a monotonic subset space. The **non-monotonic core**, denoted $\mathcal{F}^{nc}$, is a subset of $\mathcal{F}$ defined as follows:

$\mathcal{F}^{nc} = \{X \mid X \in \mathcal{F}$ and for all $X' \subseteq W$, if $X' \subseteq X$, then $X' \notin \mathcal{F}\}$.

Does every subset space $(W, \mathcal{F})$ have a non-monotonic core?

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Does every subset space $(W, \mathcal{F})$ have a non-monotonic core? No.

A monotonic collection of sets $\mathcal{F}$ is **core-complete** provided for all $X \in \mathcal{F}$, there exists a $Y \in \mathcal{F}^{nc}$ such that $Y \subseteq X$.

Observation. Suppose that $G = \langle N, \{S_i\}_{i\in N}, O, o\rangle$ is a strategic game form and E^α_G is the associated α-effectivity function. Then the non-monotonic core of $E^\alpha_G(\emptyset) = \{range(o)\}$, where $range(o) = \{x \in O \mid \text{there is a } s \in \Pi_{i\in N} S_i \text{ such that } o(s) = x\}$.

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Claim. If $E(\emptyset) = \{Y \mid Y \text{ is co-finite}\}$, then $E^{nc}(\emptyset) = \emptyset$.

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Claim. If $E(\emptyset) = \{Y \mid Y \text{ is co-finite}\}$, then $E^{nc}(\emptyset) = \emptyset$.

6. (*Empty Coalition*) $E(\emptyset)$ is core complete.

Characterizing Playable Effectivity Functions

Theorem (Pauly 2001; Goranko, Jamorga and Turrini 2013). If $E : \wp(N) \rightarrow \wp(\wp(O))$ is a function that satisfies the conditions 1-6 given above, then $E = E^{\alpha}_{G}$ for some strategic game form.

V. Goranko, W. Jamroga, and P. Turrini. *Strategic Games and Truly Playable Effectivity Functions*. Journal of Autonomous Agents and Multi-agent Systems, 26(2), pgs. 288 - 314, 2013.

M. Pauly. *Logic for Social Software*. PhD Thesis, Institute for Logic, Language and Computation, 2001.

Coalitional Models

A coalitional logic model is a tuple $\mathcal{M} = \langle W, E, V \rangle$ where W is a set of states, $E : W \rightarrow (\wp(N) \rightarrow \wp(\wp(W)))$ assigns to each state a playable effectivity function, and $V : \text{At} \rightarrow \wp(W)$ is a valuation function.

$\mathcal{M}, w \models [C]\varphi$ iff $[\![\varphi]\!]_{\mathcal{M}} = \{w \mid \mathcal{M}, w \models \varphi\} \in E(w)(C)$

Coalitional Logic: Axiomatics

1. (*Liveness*) For all $C \subseteq N$, $\emptyset \notin E(C)$

2. (*Safety*) For all $C \subseteq N$, $O \in E(C)$

3. (*N-maximality*) For all $X \subseteq O$, if $X \in E(N)$ then $\overline{X} \notin E(\emptyset)$

4. (*Outcome-monotonicity*) For all $X \subseteq X' \subseteq O$, and $C \subseteq N$, if $X \in E(C)$ then $X' \in E(C)$

5. (*Superadditivity*) For all subsets X_1, X_2 of O and sets of agents C_1, C_2, if $C_1 \cap C_2 = \emptyset$, $X_1 \in E(C_1)$ and $X_2 \in E(C_2)$, then $X_1 \cap X_2 \in E(C_1 \cup C_2)$

Coalitional Logic: Axiomatics

1. (*Liveness*) $\neg[C]\bot$

2. (*Safety*) For all $C \subseteq N$, $O \in E(C)$

3. (*N-maximality*) For all $X \subseteq O$, if $X \in E(N)$ then $\overline{X} \notin E(\emptyset)$

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4. (*Outcome-monotonicity*) $[C](\varphi \wedge \psi) \to ([C]\varphi \wedge [C]\psi)$

5. (*Superadditivity*) For all subsets X_1, X_2 of O and sets of agents C_1, C_2, if $C_1 \cap C_2 = \emptyset$, $X_1 \in E(C_1)$ and $X_2 \in E(C_2)$, then $X_1 \cap X_2 \in E(C_1 \cup C_2)$

Coalitional Logic: Axiomatics

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2. (*Safety*) $[C]\top$
3. (*N-maximality*) $[N]\varphi \rightarrow \neg[\emptyset]\neg\varphi$
4. (*Outcome-monotonicity*) $[C](\varphi \wedge \psi) \rightarrow ([C]\varphi \wedge [C]\psi)$
5. (*Superadditivity*) $([C_1]\varphi_1 \wedge [C_2]\varphi_2) \rightarrow [C_1 \cup C_2](\varphi_1 \wedge \varphi_2)$, where $C_1 \cap C_2 = \emptyset$

- ✓ Subset spaces, neighborhood frames/models, reasoning about subset spaces
- ✓ Logic of knowledge, evidence and belief
- ✓ Coalitional logic
- Interesting mathematical structures: Ultrafilters, topologies, ✓ hypergraphs

The Broader Logical Landscape

- Relational Models
- Topological Models
- *n*-ary Relational Structures
- Plausibility Structures
- First-Order Logic

From Kripke Frames to Neighborhood Frames

Let $R \subseteq W \times W$, define a map $R^{\rightarrow} : W \rightarrow \wp W$:

$$\text{for each } w \in W, \text{ let } R^{\rightarrow}(w) = \{v \mid wRv\}$$

From Kripke Frames to Neighborhood Frames

Let $R \subseteq W \times W$, define a map $R^{\rightarrow} : W \rightarrow \wp W$:

$$\text{for each } w \in W, \text{ let } R^{\rightarrow}(w) = \{v \mid wRv\}$$

Definition

Given a relation R on a set W and a state $w \in W$. A set $X \subseteq W$ is *R-necessary at w* if $R^{\rightarrow}(w) \subseteq X$.

From Kripke Frames to Neighborhood Frames

Let $R \subseteq W \times W$, define a map $R^{\rightarrow} : W \rightarrow \wp W$:

$$\text{for each } w \in W, \text{ let } R^{\rightarrow}(w) = \{v \mid wRv\}$$

Let $\mathcal{N}_w^R$ be the set of sets that are R-necessary at w:

$$\mathcal{N}_w^R = \{X \mid R^{\rightarrow}(w) \subseteq X\}$$

From Kripke Frames to Neighborhood Frames

Let $R \subseteq W \times W$, define a map $R^{\rightarrow} : W \rightarrow \wp W$:

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Let $\mathcal{N}_w^R$ be the set of sets that are R-necessary at w:

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Lemma

Let R be a relation on W. Then for each $w \in W$, $\mathcal{N}_w^R$ is augmented.

From Kripke Frames to Neighborhood Frames

Properties of R are reflected in $\mathcal{N}^R_w$:

- If R is reflexive, then for each $w \in W$, $w \in \cap\mathcal{N}_w$
- If R is transitive then for each $w \in W$, if $X \in \mathcal{N}_w$, then $\{v \mid X \in \mathcal{N}_v\} \in \mathcal{N}_w$.

From Neighborhood Frames to Kripke Frames

Theorem

- *Let $\langle W, R\rangle$ be a relational frame. Then there is an equivalent augmented neighborhood frame.*

- *Let $\langle W, N\rangle$ be an augmented neighborhood frame. Then there is an equivalent relational frame.*

From Neighborhood Frames to Kripke Frames

for all $X \subseteq W$, $X \in N(w)$ iff $X \in \mathcal{N}_w^R$.

Theorem

- *Let $\langle W, R \rangle$ be a relational frame. Then there is an equivalent augmented neighborhood frame.*
- *Let $\langle W, N \rangle$ be an augmented neighborhood frame. Then there is an equivalent relational frame.*

From Neighborhood Frames to Kripke Frames

Theorem

- ✓ *Let $\langle W, R \rangle$ be a relational frame. Then there is an equivalent augmented neighborhood frame.*

- *Let $\langle W, N \rangle$ be an augmented neighborhood frame. Then there is an equivalent relational frame.*

Proof.
For each $w \in W$, let $N(w) = \mathcal{N}_w^R$. □

From Neighborhood Frames to Kripke Frames

Theorem

- *Let $\langle W, R\rangle$ be a relational frame. Then there is an equivalent augmented neighborhood frame.*

✓ *Let $\langle W, N\rangle$ be an augmented neighborhood frame. Then there is an equivalent relational frame.*

Proof.

For each $w, v \in W$, $wR_N v$ iff $v \in \cap N(w)$. □

Topological Models for Modal Logic

Definition

Topological Space A **topological space** is a neighborhood frame $\langle W, \mathcal{T} \rangle$ where W is a nonempty set and

1. $W \in \mathcal{T}$, $\emptyset \in W$
2. $\mathcal{T}$ is closed under finite intersections
3. $\mathcal{T}$ is closed under arbitrary unions.

Topological Models for Modal Logic

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2. $\mathcal{T}$ is closed under finite intersections
3. $\mathcal{T}$ is closed under arbitrary unions.

A neighborhood of w is any set X such that there is an $O \in \mathcal{T}$ with $w \in O \subseteq N$

Let $\mathcal{T}_w$ be the collection of all neighborhoods of w.

Topological Models for Modal Logic

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1. $W \in \mathcal{T}$, $\emptyset \in W$
2. $\mathcal{T}$ is closed under finite intersections
3. $\mathcal{T}$ is closed under arbitrary unions.

Lemma

Let $\langle W, \mathcal{T} \rangle$ be a topological space. Then for each $w \in W$, the collection $\mathcal{T}_w$ contains W, is closed under finite intersections and closed under arbitrary unions.

Topological Models for Modal Logic

The largest open subset of X is called the interior of X, denoted $Int(X)$. Formally,

$$Int(X) = \cup\{O \mid O \in \mathcal{T} \text{ and } O \subseteq X\}$$

The smallest closed set containing X is called the closure of X, denoted $Cl(X)$. Formally,

$$Cl(X) = \cap\{C \mid W - C \in \mathcal{T} \text{ and } X \subseteq C\}$$

Topological Models for Modal Logic

- $Int(X) = \cup\{O \mid O \in \mathcal{T} \text{ and } O \subseteq X\}$
- $Cl(X) = \cap\{C \mid W - C \in \mathcal{T} \text{ and } X \subseteq C\}$

Lemma

Let $\langle W, \mathcal{T}\rangle$ be a topological space and $X \subseteq W$. Then

1. $Int(X \cap Y) = Int(X) \cap Int(Y)$
2. $Int(\emptyset) = \emptyset$, $Int(W) = W$
3. $Int(X) \subseteq X$
4. $Int(Int(X)) = Int(X)$
5. $Int(X) = W - Cl(W - X)$

Topological Models for Modal Logic

- $Int(X) = \cup\{O \mid O \in \mathcal{T} \text{ and } O \subseteq X\}$
- $Cl(X) = \cap\{C \mid W - C \in \mathcal{T} \text{ and } X \subseteq C\}$

Lemma

Let $\langle W, \mathcal{T}\rangle$ be a topological space and $X \subseteq W$. Then

1. $\Box(\varphi \wedge \psi) \leftrightarrow \Box\varphi \wedge \Box\psi$
2. $\Box\bot \leftrightarrow \bot, \Box\top \leftrightarrow \top$
3. $\Box\varphi \rightarrow \varphi$
4. $\Box\Box\varphi \leftrightarrow \Box\varphi$
5. $\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$

Topological Models for Modal Logic

A topological model is a triple $\langle W, \mathcal{T}, V\rangle$ where $\langle W, \mathcal{T}\rangle$ is a topological space and V a valuation function.

Topological Models for Modal Logic

A topological model is a triple $\langle W, \mathcal{T}, V\rangle$ where $\langle W, \mathcal{T}\rangle$ is a topological space and V a valuation function.

$$\mathbb{M}^T, w \models \Box\varphi \text{ iff } \exists O \in \mathcal{T}, w \in O \text{ such that } \forall v \in O, \mathbb{M}^T, v \models \varphi$$

$$(\Box\varphi)^{\mathbb{M}^T} = Int((\varphi)^{\mathbb{M}^T})$$

From Neighborhoods to Topologies

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A family $\mathcal{B}$ of subsets of W is called a basis for a topology $\mathcal{T}$ if every open set can be represented as the union of elements of a subset of $\mathcal{B}$

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A family $\mathcal{B}$ of subsets of W is called a basis for a topology $\mathcal{T}$ if every open set can be represented as the union of elements of a subset of $\mathcal{B}$

Fact: A family $\mathcal{B}$ of subsets of W is a basis for some topology if

- for each $w \in W$ there is a $U \in \mathcal{B}$ such that $w \in U$
- for each $U, V \in \mathcal{B}$, if $w \in U \cap V$ then there is a $W \in \mathcal{B}$ such that $w \in W \subseteq U \cap V$

From Neighborhoods to Topologies

A family $\mathcal{B}$ of subsets of W is called a basis for a topology $\mathcal{T}$ if every open set can be represented as the union of elements of a subset of $\mathcal{B}$

Let $\mathbb{M} = \langle W, N, V \rangle$ be a neighborhood models. Suppose that N satisfies the following properties

- for each $w \in W$, $N(w)$ is a filter
- for each $w \in W$, $w \in \cap N(w)$
- for each $w \in W$ and $X \subseteq W$, if $X \in N(w)$, then $m_N(X) \in N(w)$

Then there is a topological model that is point-wise equivalent to $\mathbb{M}$.

J. van Benthem and G. Bezhanishvili. *Modal Logics of Space*. Handbook of Spatial Logics, pgs. 217 - 298, 2007.

Generalized Relational Models

- *n*-ary relations
- multiple relations
- non-normal worlds

n-ary Relations

$$(\Box\varphi \wedge \Box\psi) \rightarrow \Box(\varphi \wedge \psi)$$

n-ary Relations

$$(\Box\varphi \wedge \Box\psi) \rightarrow \Box(\varphi \wedge \psi)$$

An n-**ary relational model** is a tuple $\langle W, R, V\rangle$ where W is a non-empty set and $R \subseteq W^n$ is an n-ary relation ($R \subseteq W^n$) and $V : \mathsf{At} \rightarrow \wp(W)$ is a valuation function. (Assume $n \geq 2$)

n-ary Relations

$$(\Box\varphi \wedge \Box\psi) \rightarrow \Box(\varphi \wedge \psi)$$

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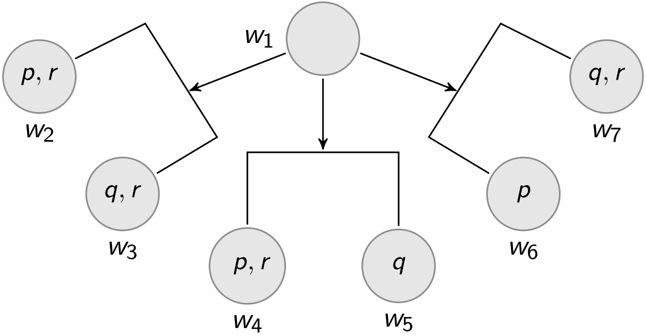
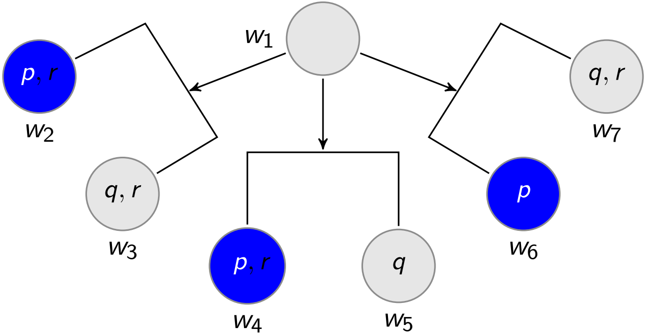
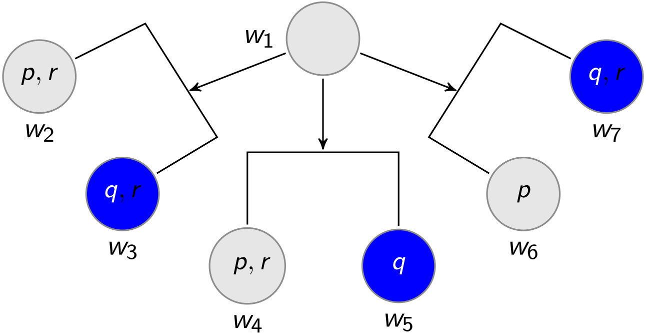
- $\mathcal{M}^n, w \models \Box\varphi$ iff for all $(w_1, \ldots, w_{n-1}) \in W^{n-1}$, if $(w, w_1, \ldots, w_n) \in R$, then there exists i such that $1 \leq i \leq n$ and $\mathcal{M}^n, w_i \models \varphi$.
- $\mathcal{M}^n, w \models \Diamond\varphi$ iff there exists $(w_1, \ldots, w_n) \in W^{n-2}$ such that $(w, w_1, \ldots, w_n) \in R$, and for all i such that $1 \leq i \leq n$, we have $\mathcal{M}^n, w_i \models \varphi$.

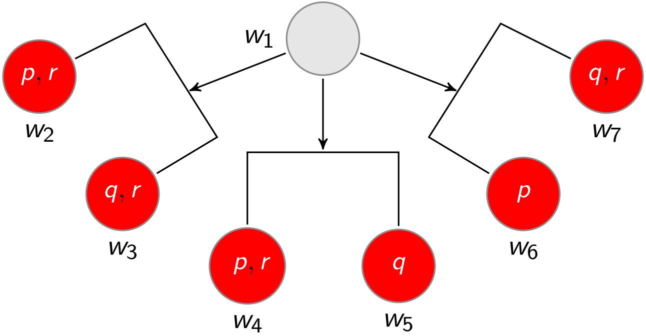

w_1
p, r
w_2
q, r
w_3
p, r
w_4
q
w_5
p
w_6
q, r
w_7



- $\mathcal{M}^3, w_1 \models \Box p$ (and $\mathcal{M}^3, w_1 \models \Box\neg p$)



- $\mathcal{M}^3, w_1 \models \Box p$ (and $\mathcal{M}^3, w_1 \models \Box\neg p$)
- $\mathcal{M}^3, w_1 \models \Box q$ (and $\mathcal{M}^3, w_1 \models \Box\neg q$)



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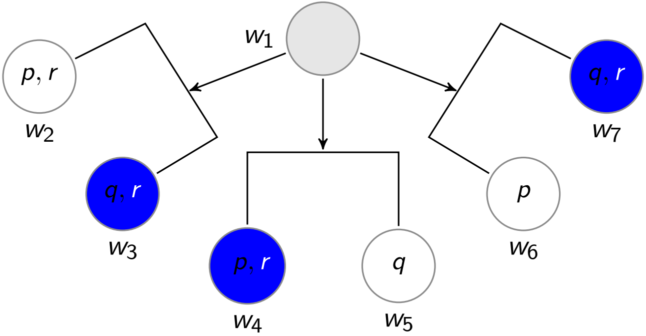
- $\mathcal{M}^3, w_1 \not\models \Box(p \wedge q)$



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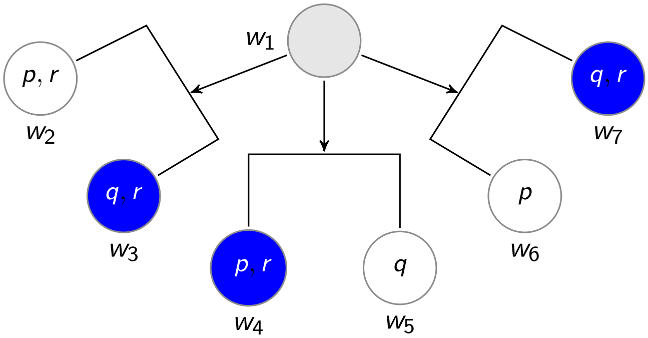
- $\mathcal{M}^3, w_1 \models \Box r$



- $\mathcal{M}^3, w_1 \models \Box p$ (and $\mathcal{M}^3, w_1 \models \Box\neg p$)
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- $\mathcal{M}^3, w_1 \not\models \Box(p \wedge q)$
- $\mathcal{M}^3, w_1 \models \Box r$
- $\mathcal{M}^3, w_1 \models \Box((p \wedge r) \vee (q \wedge r))$

$$(C^n) \qquad \bigwedge_{i=1}^{n} \Box\varphi_i \rightarrow \Box \bigvee_{1\leq k,l\leq n, k\neq l} (\varphi_k \wedge \varphi_l)$$

$$(C^n) \qquad \bigwedge_{i=1}^{n} \Box\varphi_i \to \Box \bigvee_{1\leq k,l\leq n, k\neq l} (\varphi_k \wedge \varphi_l)$$

Example:
$(\Box\varphi_1 \wedge \Box\varphi_2 \wedge \Box\varphi_3) \to \Box((\varphi_1 \wedge \varphi_2) \vee (\varphi_2 \wedge \varphi_3) \vee (\varphi_1 \wedge \varphi_3))$

Suppose that $\mathbf{L}(\mathfrak{C}^n) = \{\varphi \in \mathcal{L}(\mathsf{At}) \mid \text{for all } \mathcal{F}^n \in \mathfrak{C}^n, \mathcal{F}^n \models \varphi\}$.

$$\mathbf{EMN} = \bigcap_{n \geq 2} \mathbf{L}(\mathfrak{C}^n)$$

Theorem. The logic $\mathbf{EMNC}^n$ is sound and complete for the class $\mathfrak{C}^n$ of n-ary relational frames.

Proposition. Suppose that $\mathcal{M} = \langle W, N, V \rangle$ is finite monotonic neighborhood model such that for all $w \in W$, $N(w) \neq \emptyset$. Then, there is an *n*-ary relational model $\mathcal{M}^N = \langle W^N, R^N, V^N \rangle$ that is modally equivalent to $\mathcal{M}$.

Proposition. Suppose that $\mathcal{M}^n = \langle W, R, V \rangle$ is a finite *n*-ary relational model. Then, there is a finite monotonic neighborhood model $\mathcal{M}^R = \langle W^R, N^R, V^R \rangle$ that is modally equivalent to $\mathcal{M}^n$.

Multi-Relational Semantics/Non-Normal Modal Logics

A multi-relational Kripke model is a triple $\mathbb{M} = \langle W, \mathcal{R}, V \rangle$ where $\mathcal{R} \subseteq \wp(W \times W)$.

Multi-Relational Semantics/Non-Normal Modal Logics

A multi-relational Kripke model is a triple $\mathbb{M} = \langle W, \mathcal{R}, V \rangle$ where $\mathcal{R} \subseteq \wp(W \times W)$.

$\mathbb{M}, w \models \Box\varphi$ iff $\exists R \in \mathcal{R}$ such that $\forall v \in W$, if wRv then $\mathbb{M}, v \models \varphi$.

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Are multi-relational semantics *equivalent* to neighborhood semantics?

Multi-Relational Semantics/Non-Normal Modal Logics

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Are multi-relational semantics *equivalent* to neighborhood semantics? Almost

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$\mathbb{M}, w \models \Box\varphi$ iff $\exists R \in \mathcal{R}$ such that $\forall v \in W$, if wRv then $\mathbb{M}, v \models \varphi$.

A world is called impossible if nothing is necessary and everything is possible.

Multi-Relational Semantics/Non-Normal Modal Logics

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w is an impossible world iff $N(w) = \emptyset$

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w is an impossible world iff $N(w) = \emptyset$

A multi-relational model with impossible worlds is a quadruple $\mathbb{M} = \langle W, Q, \mathcal{R}, V \rangle$.

Multi-Relational Semantics/Non-Normal Modal Logics

A multi-relational Kripke model is a triple $\mathbb{M} = \langle W, \mathcal{R}, V \rangle$ where $\mathcal{R} \subseteq \wp(W \times W)$.

$\mathbb{M}, w \models \Box\varphi$ iff $\exists R \in \mathcal{R}$ such that $\forall v \in W$, if wRv then $\mathbb{M}, v \models \varphi$.

w is an impossible world iff $N(w) = \emptyset$

A multi-relational model with impossible worlds is a quadruple $\mathbb{M} = \langle W, Q, \mathcal{R}, V \rangle$.

$\mathbb{M}, w \models \Box\varphi$ iff $w \not\in Q$ and $\exists R \in \mathcal{R}$ such that $\forall v \in W$, if wRv then $\mathbb{M}, v \models \varphi$.

Multi-Relational Semantics/Non-Normal Modal Logics

M. Fitting. *Proof Methods for Modal and Intuitionistic Logics*. Synthese Library, 1983.

L. Goble. *Multiplex semantics for Deontic Logic*. Nordic Journal of Philosophical Logic, 5(2), pgs. 113-134, 2000.

G. Governatori and A. Rotolo. *On the axiomatization of Elgesem's logic of agency and ability*. Journal of Philosophical Logic, 34(4), pgs. 403 - 431, 2005.

Let $Th_{\mathcal{L}}(\mathcal{M}, w) = \{\varphi \in \mathcal{L} \mid \mathcal{M}, w \models \varphi\}$

Suppose that M and M$'$ are two classes of models for $\mathcal{L}$. Say that $\mathcal{M}, w$ is $\mathcal{L}$-equivalent to $\mathcal{M}', w'$, denoted $\mathcal{M}, w \equiv_{\mathcal{L}} \mathcal{M}', w'$, provided $Th_{\mathcal{L}}(\mathcal{M}, w) = Th_{\mathcal{L}}(\mathcal{M}', w')$.

A class of models M is $\mathcal{L}$-equivalent to a class of models M$'$ provided for each pointed model $\mathcal{M}, w$ from M, there exists a pointed model $\mathcal{M}', w'$ from M$'$ such that $\mathcal{M}, w \equiv_{\mathcal{L}} \mathcal{M}', w'$, and *vice versa*.

- The class $\mathsf{K} = \{\mathcal{M} \mid \mathcal{M}$ is a relational model $\}$ is modally equivalent to the class
  $\mathsf{M}_{aug} = \{\mathcal{M} \mid \mathcal{M}$ is an augmented neighborhood model$\}$

- The class $\mathsf{K}^n = \{\mathcal{M}^n \mid \mathcal{M}$ is an n-ary relational model $\}$ is modally equivalent to the class
  $\mathsf{M}_{reg} = \{\mathcal{M} \mid \mathcal{M}$ is a consistent regular neighborhood model$\}$

- The class $\mathsf{T} = \{\mathcal{M}^T \mid \mathcal{M}$ is a topological model $\}$ is modally equivalent to the class
  $\mathsf{M}_{\mathsf{S4}} = \{\mathcal{M} \mid \mathcal{M}$ is an **S4** neighborhood model$\}$

Core Theory

- ✓ Neighborhood Semantics in the Broader Logical Landscape
- Completeness, Decidability, Complexity
- Incompleteness
- Relation with Relational Semantics
- Model Theory

Useful Fact

Theorem (Uniform Substitution)

The following rule can be derived in **E**

$$\frac{\psi \leftrightarrow \psi'}{\varphi \leftrightarrow \varphi[\psi/\psi']}$$

Interesting Fact

Each of K, M and C are logically independent:

- $\mathbf{EC} \nvdash K$
- $\mathbf{EM} \nvdash K$
- $\mathbf{EMC} \vdash K$
- $\mathbf{EK} \nvdash M$
- $\mathbf{EK} \nvdash C$

Neighborhood Semantics for Modal Logic

Eric Pacuit

Email: epacuit@umd.edu

Web: pacuit.org

Lecture 3: June 7th, 14h00-16h30