Neighborhood Semantics for Modal Logic
Lecture 1

Eric Pacuit

University of Maryland, College Park
pacuit.org
epacuit@umd.edu

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Text

Neighborhood semantics for modal logic (Draft)

Ch 1: Introduction and Motivation

Ch 2: Core Theory: Expressivity, Completeness, Decidability, Complexity, Correspondence Theory

Ch 3: Richer Languages: Fixed-point operators, First-order extensions, Dynamic operators

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Ch 1: Introduction and Motivation

Ch 2: Core Theory: Expressivity, Completeness, Decidability, Complexity, Correspondence Theory

Ch 3: Richer Languages: Fixed-point operators, First-order extensions, Dynamic operators

Current research papers non-normal modal logics and neighborhood structures.

Prerequisites

There are no specific prerequisites, although some background on modal logic will be helpful.

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Appendix A: Relational Models for Modal Logic

Schedule

Lecture 1: June 1st, 14h00-16h30

Lecture 2: June 2nd 12h30-14h30

Lecture 3: June 7th, 14h00-16h30

Lecture 4: June 8th, 11h00-13h00

Lecture 5: June 8th, 14h00-16h30

Lecture 7: June 9th, 12h30-14h30

Lecture 8: June 13th,12h30-15h00

Lecture 9: June 14th, 10h00-13h00

Lecture 10 Presentations (solutions to problems etc.): June 15th, 10h00-13h00

Location: F1.15 ILLC

1. Non-normal modal logics
2. Neighborhood semantics for modal logic

Normal modal logic

The Basic Modal Language: $\mathcal{L}$

$$p \mid \neg\varphi \mid \varphi \wedge \psi \mid \Box\varphi \mid \Diamond\varphi$$

where p is an atomic proposition (Let At be the set of atomic propositions)

One Language, Many Interpretations

tense: henceforth, eventually, previously, now, tomorrow, yesterday, since, until, it will have been, it is being,. . .

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geometric: it is locally the case that

metalogic: it is valid/satisfiable/provable/consistent that

game/action: there exist a strategy/action to guarantee that

Relational Structures

Relational (Kripke) Frame: $\langle W, R \rangle$

- $W \neq \emptyset$
- $R \subseteq W \times W$

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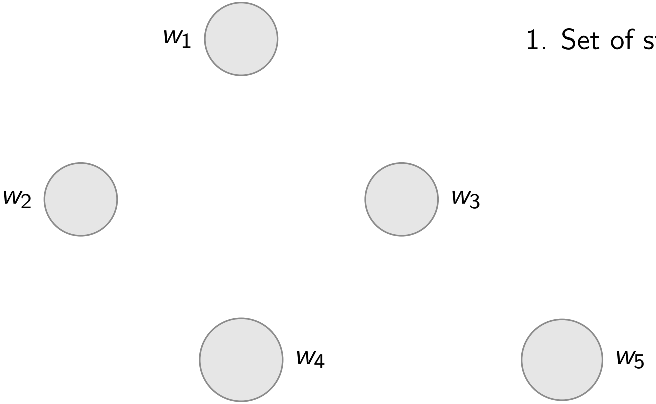
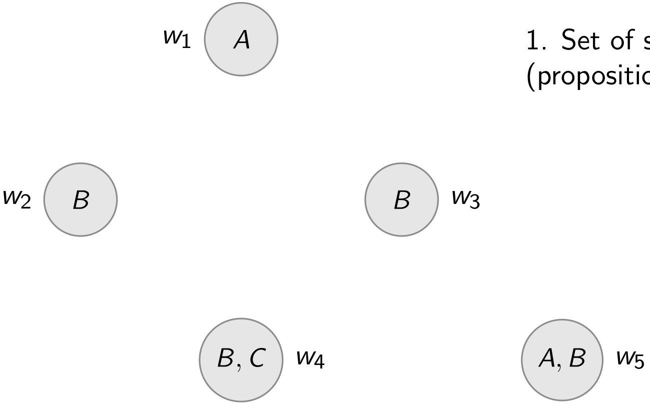
- $W \neq \emptyset$
- $R \subseteq W \times W$

Relational (Kripke) Model: $\langle W, R, V\rangle$

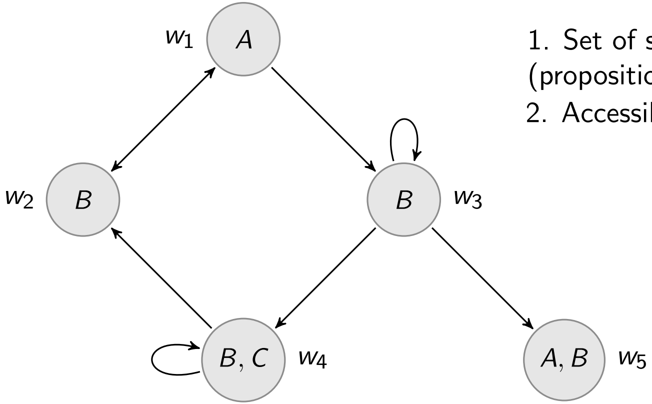
- $\langle W, R\rangle$ is a frame
- $V : \mathsf{At} \rightarrow \wp(W)$



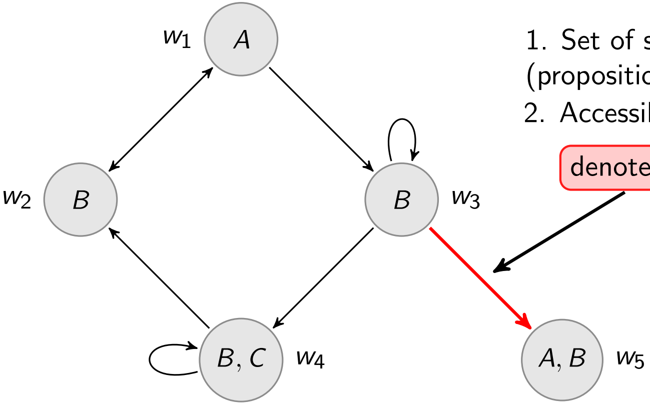
1. Set of states



1. Set of states
(propositional valuations)



1. Set of states (propositional valuations)
2. Accessibility relation



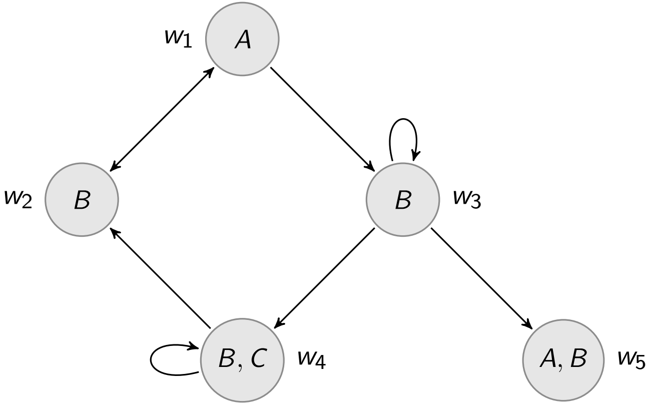
1. Set of states (propositional valuations)
2. Accessibility relation

denoted $w_3 R w_5$

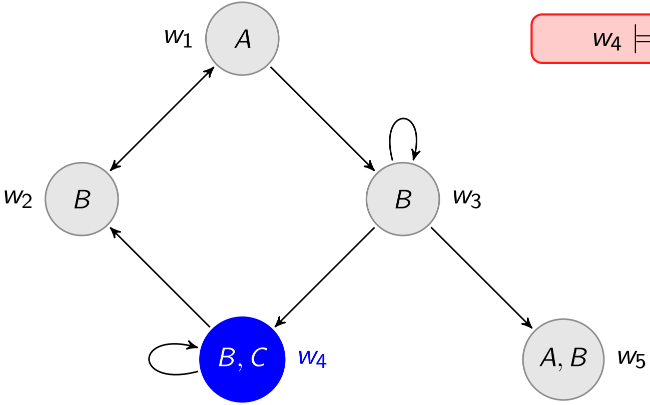
Truth: $\mathcal{M}, w \models \varphi$

1. $\mathcal{M}, w \models p$ iff $w \in V(p)$
2. $\mathcal{M}, w \models \neg\varphi$ iff $\mathcal{M}, w \not\models \varphi$
3. $\mathcal{M}, w \models \varphi \wedge \psi$ iff $\mathcal{M}, w \models \varphi$ and $\mathcal{M}, w \models \psi$
4. $\mathcal{M}, w \models \Box\varphi$ iff for each $v \in W$, if wRv then $\mathcal{M}, v \models \varphi$
5. $\mathcal{M}, w \models \Diamond\varphi$ iff there is a $v \in W$ such that wRv and $\mathcal{M}, v \models \varphi$

Example

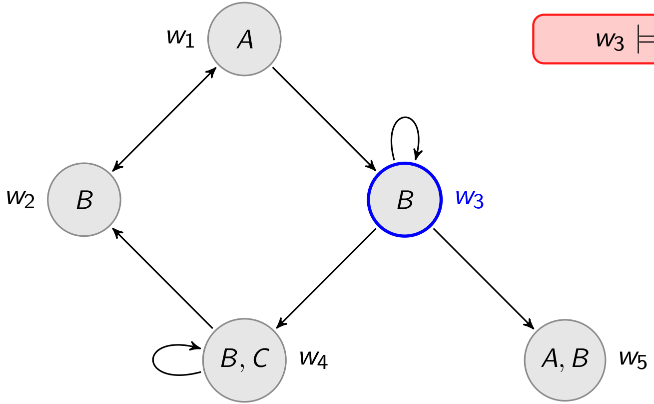


Example



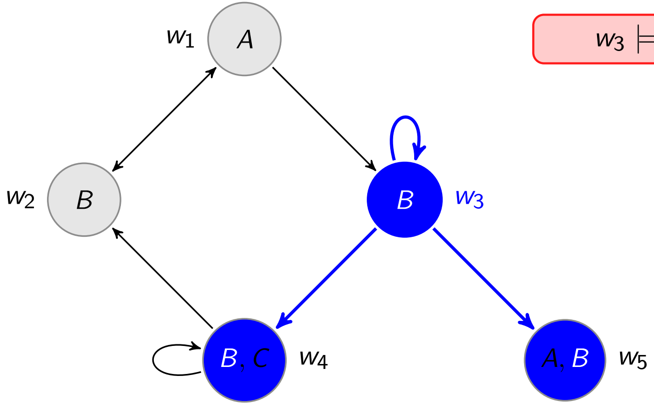
$w_4 \models B \wedge C$

Example



$w_3 \models \Box B$

Example



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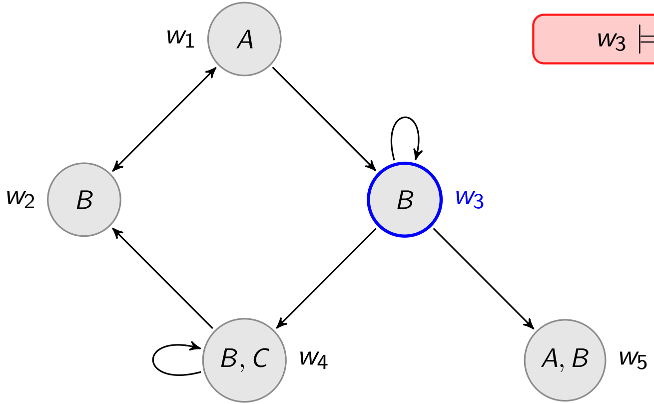
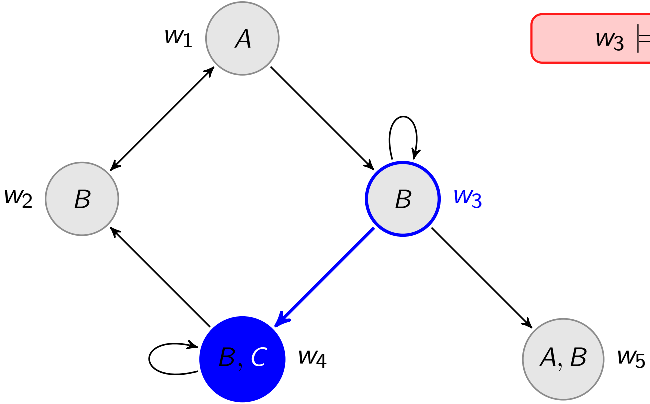
Example



$w_3 \models \Diamond C$

Example



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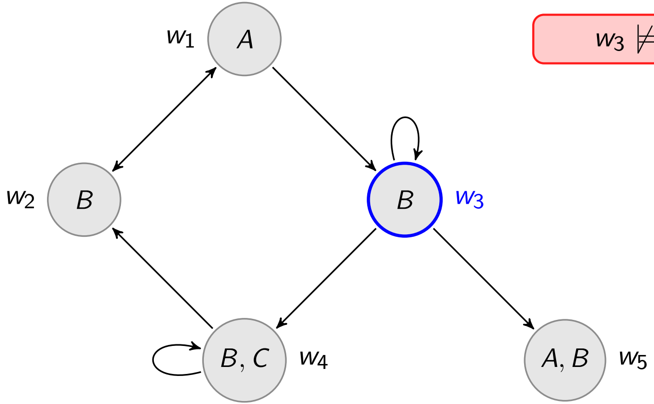
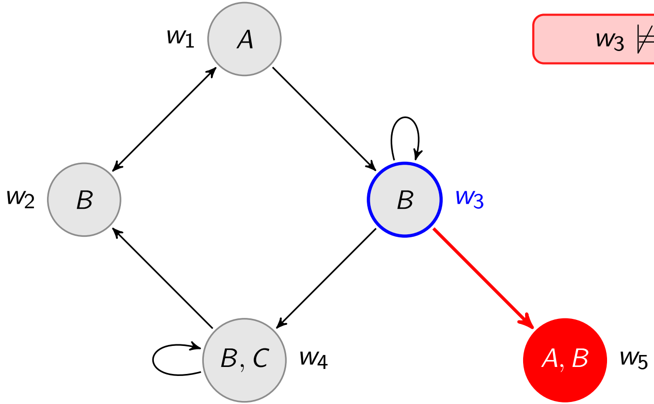


$w_3 \not\models \Box C$

Example



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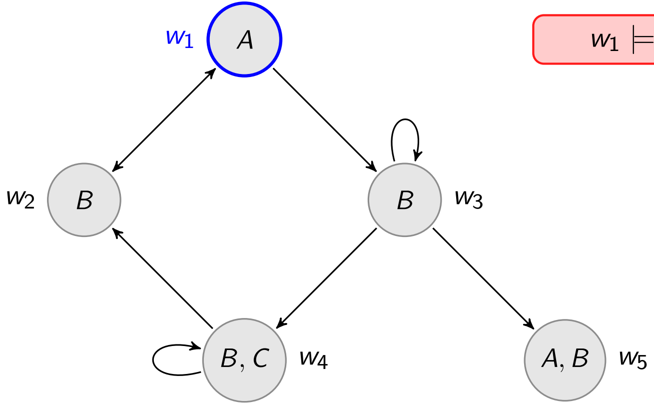
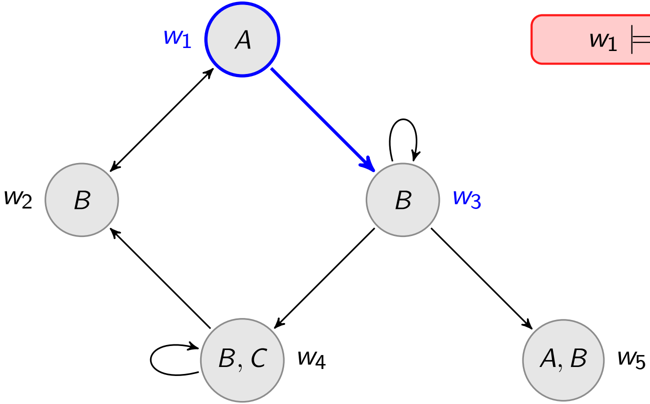


$w_1 \models \Diamond\Box B$

Example



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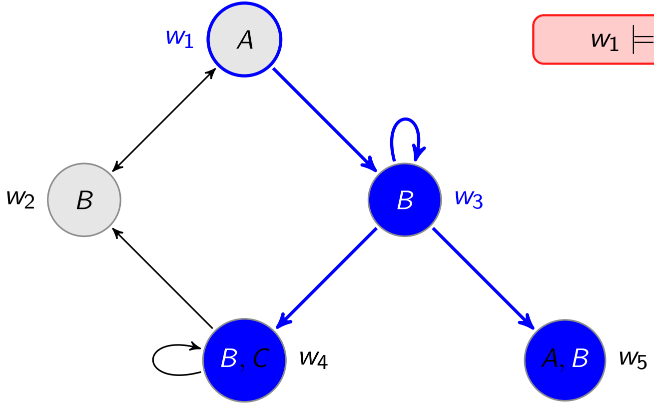
Example



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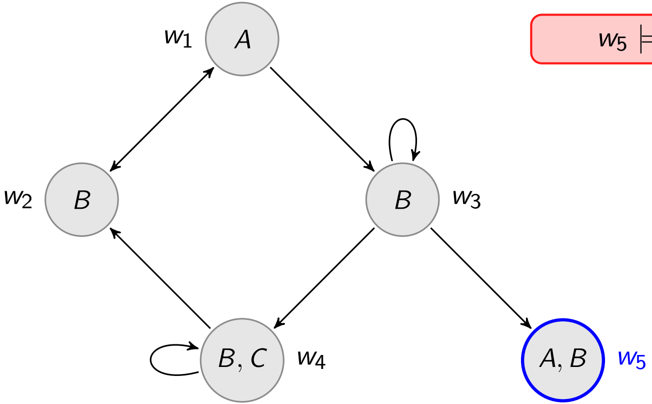
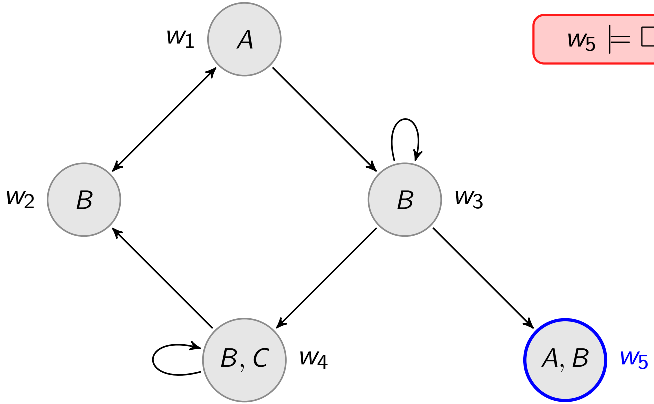
Example



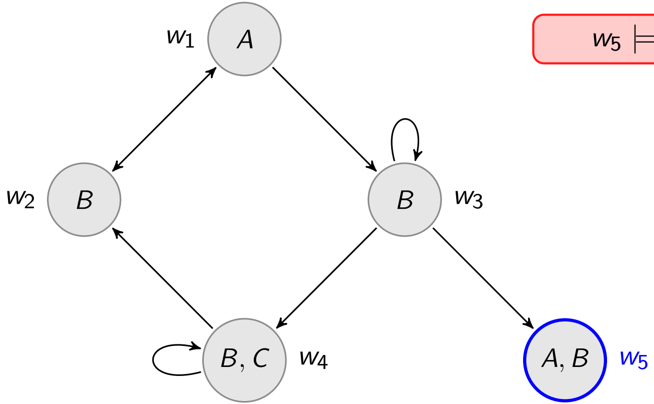
$w_5 \models \Box C$

Example



$w_5 \models \Box(B \wedge \neg B)$

Example

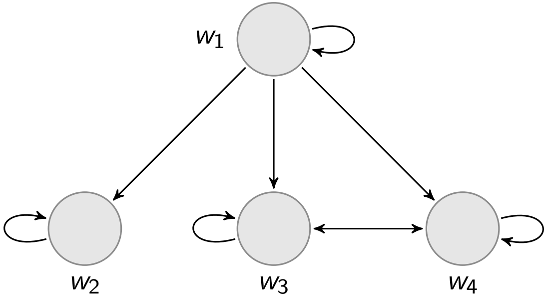


$w_5 \models \neg\Diamond B$

Standard Logical Notions

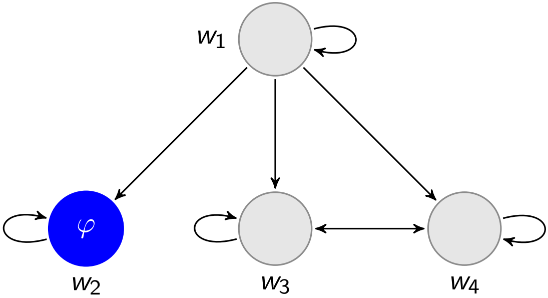
Valid on a model:	$\mathcal{M} \models \varphi$
Valid at a state on a frame:	$\mathcal{F}, w \models \varphi$
Valid on a frame:	$\mathcal{F} \models \varphi$
Valid in a class F of frame:	$\models_{\mathsf{F}} \varphi$

Example



$$\mathcal{F}, w_1 \models \Box\Diamond\varphi \rightarrow \Diamond\Box\varphi$$

Example



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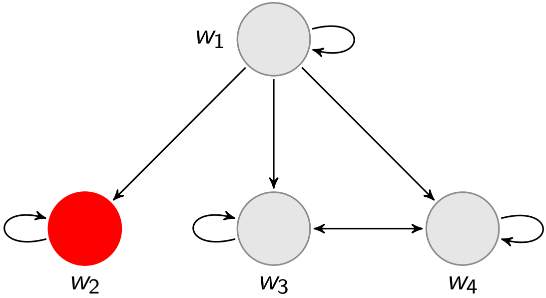

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Some Validities

(M)	$\Box(\varphi \wedge \psi) \rightarrow \Box\varphi \wedge \Box\psi$
(C)	$\Box\varphi \wedge \Box\psi \rightarrow \Box(\varphi \wedge \psi)$
(N)	$\Box\top$
(K)	$\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$
(Dual)	$\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$
(Nec)	from $\vdash \varphi$ infer $\vdash \Box\varphi$
(Re)	from $\vdash \varphi \leftrightarrow \psi$ infer $\vdash \Box\varphi \leftrightarrow \Box\psi$

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(Nec) from $\vdash \varphi$ infer $\vdash \Box\varphi$

(Re) from $\vdash \varphi \leftrightarrow \psi$ infer $\vdash \Box\varphi \leftrightarrow \Box\psi$

(*RM*) $$\frac{\vdash \varphi \rightarrow \psi}{\vdash \Box\varphi \rightarrow \Box\psi}$$

The History of Modal Logic

R. Goldblatt. *Mathematical Modal Logic: A View of its Evolution*. Handbook of the History of Logic, Vol. 7, 2006.

P. Balckburn, M. de Rijke, and Y. Venema. *Modal Logic*. Section 1.7, Cambridge University Press, 2001.

R. Ballarin. *Modern Origins of Modal Logic*. Stanford Encyclopedia of Philosophy, 2010.

Non-normal modal logics

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Non-normal modal logics

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(Non-)Normal Modal Logic

Let $\mathcal{L}$ be the basic modal language.

A **modal logic** is a set of formulas from $\mathcal{L}$. If $\mathbf{L}$ is a modal logic, then we write $\vdash_{\mathbf{L}} \varphi$ when $\varphi \in \mathbf{L}$.

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A modal logic $\mathbf{L}$ is **normal** provided $\mathbf{L}$ is

- contains propositional logic (i.e., all instances of the propositional axioms and closed under Modus Ponens)
- closed under Necessitation (from $\vdash_{\mathbf{L}} \varphi$ infer $\vdash_{\mathbf{L}} \Box\varphi$);
- contains all instances of K ($\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$); and
- closed under *uniform substitution*.

(Non-)Normal Modal Logic

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Normal Modal Logic

The smallest normal modal logic **K** consists of

PC Your favorite axioms of **PC**

K $\Box(\varphi \to \psi) \to \Box\varphi \to \Box\psi$

Nec $\dfrac{\vdash \varphi}{\Box\varphi}$

MP $\dfrac{\vdash \varphi \to \psi \qquad \vdash \varphi}{\psi}$

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Theorem. K is sound and strongly complete with respect to the class of all relational frames.

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Theorem. For all $\Gamma \subseteq \mathcal{L}$, $\Gamma \vdash_{\mathbf{K}} \varphi$ iff $\Gamma \models \varphi$.

Normal Modal Logic

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PC Your favorite axioms of **PC**

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MP $\dfrac{\vdash \varphi \to \psi \qquad \vdash \varphi}{\psi}$

Theorem. **K** $+ \Box\varphi \to \varphi + \Box\varphi \to \Box\Box\varphi$ is sound and strongly complete with respect to the class of all reflexive and transitive relational frames.

Are there non-normal extensions of **K**?

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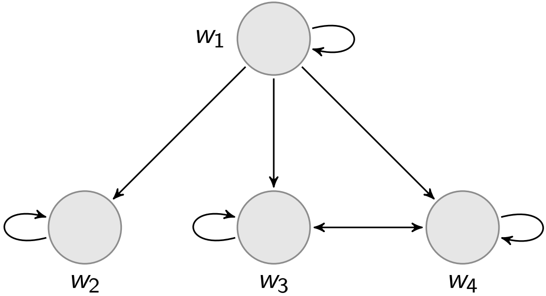
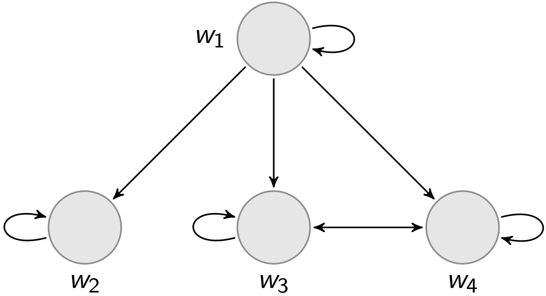
Let **L** be the smallest modal logic containing

- **S4** (**K** $+\ \Box\varphi \to \varphi\ +\ \Box\varphi \to \Box\Box\varphi$)
- all instances of M: $\Box\Diamond\varphi \to \Diamond\Box\varphi$

Claim: **L** is a non-normal extension of **S4**.

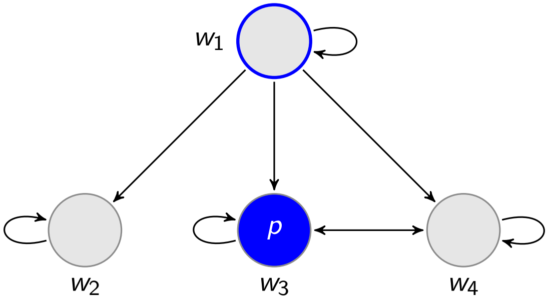


$\mathcal{F}, w_1 \models \Box\Diamond\varphi \rightarrow \Diamond\Box\varphi$



$$\mathcal{F}, w_1 \models \Box\Diamond\varphi \rightarrow \Diamond\Box\varphi$$

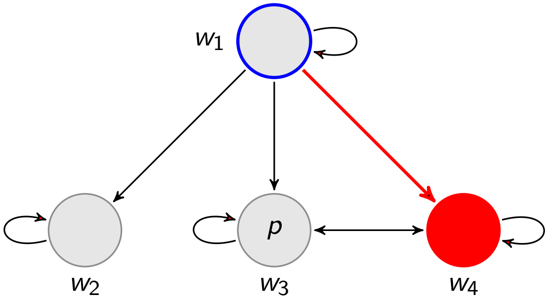
$$\mathbf{L} \subseteq \mathbf{L}_{w_1} = \{\varphi \mid \mathcal{F}, w_1 \models \varphi\}$$



$\mathcal{F}, w_1 \models \Box\Diamond\varphi \rightarrow \Diamond\Box\varphi$

$\mathbf{L} \subseteq \mathbf{L}_{w_1} = \{\varphi \mid \mathcal{F}, w_1 \models \varphi\}$

$\mathcal{F}, w_1 \not\models \Box(\Box\Diamond p \rightarrow \Diamond\Box p)$



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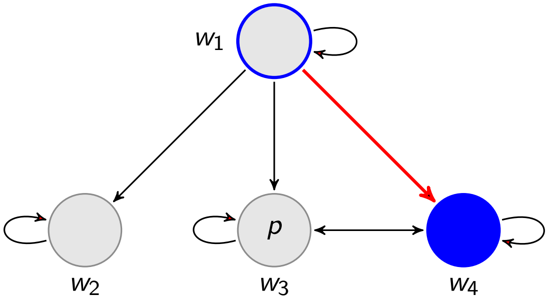

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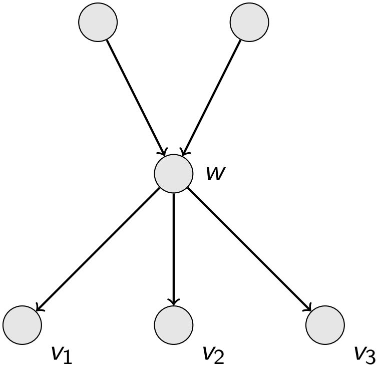
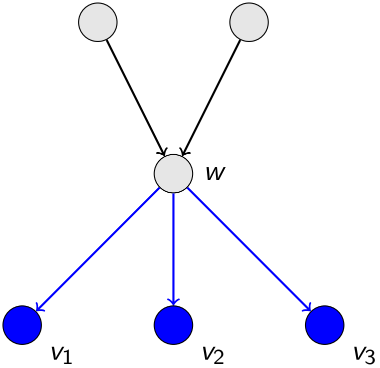
✓ Non-normal modal logics

1. Neighborhood semantics for modal logic


w
v1
v2
v3



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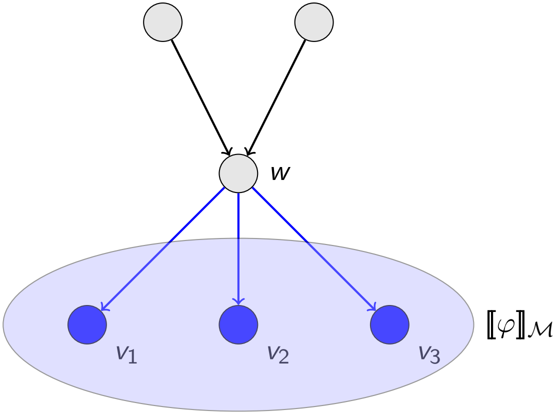
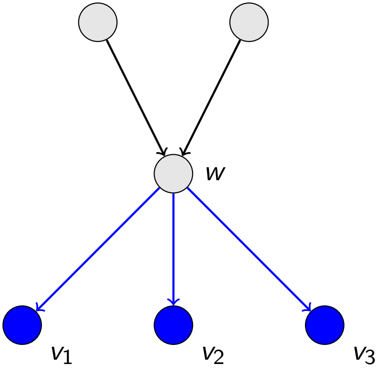


$\mathcal{M}, w \models \Box\varphi$ iff $R(w) \subseteq [\![\varphi]\!]_{\mathcal{M}}$

...**the** neighborhood of w is
contained in the truth-set of φ



$$\mathcal{M}, w \models \boxplus\varphi \text{ iff } R(w) = [\![\varphi]\!]_{\mathcal{M}}$$

...**the** neighborhood of w **is** the truth-set of φ

Neighborhoods in Topology

In a topology, a *neighborhood* of a point x is any set A containing x such that you can "wiggle" x without leaving A.

A *neighborhood system* of a point x is the collection of neighborhoods of x.

J. Dugundji. *Topology*. 1966.

$w \models \Box\varphi$ if the truth set of φ is a neighborhood of w

$w \models \Box\varphi$ if the truth set of φ is a neighborhood of w

What does it mean to be a neighborhood?

$w \models \Box\varphi$ if the truth set of φ is a neighborhood of w

neighborhood in some topology.

J. McKinsey and A. Tarski. *The Algebra of Topology*. 1944.

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J. McKinsey and A. Tarski. *The Algebra of Topology*. 1944.

contains all the immediate neighbors in some graph

S. Kripke. *A Semantic Analysis of Modal Logic*. 1963.

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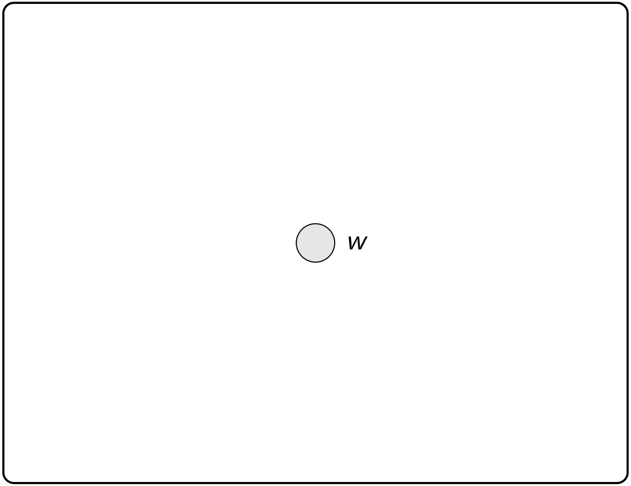
S. Kripke. *A Semantic Analysis of Modal Logic*. 1963.

an element of some distinguished collection of sets

D. Scott. *Advice on Modal Logic*. 1970.

R. Montague. *Pragmatics*. 1968.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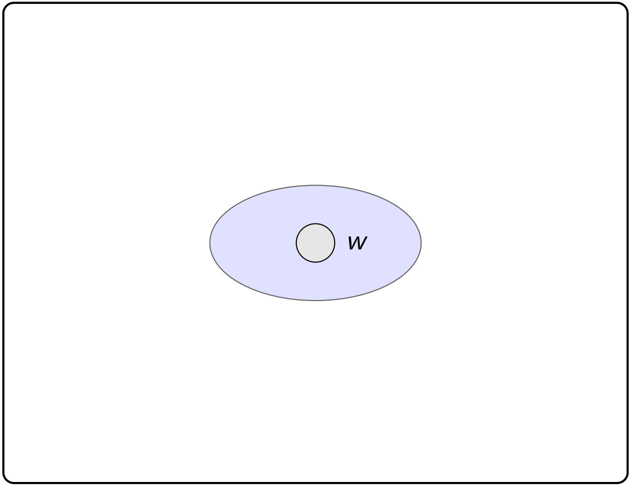
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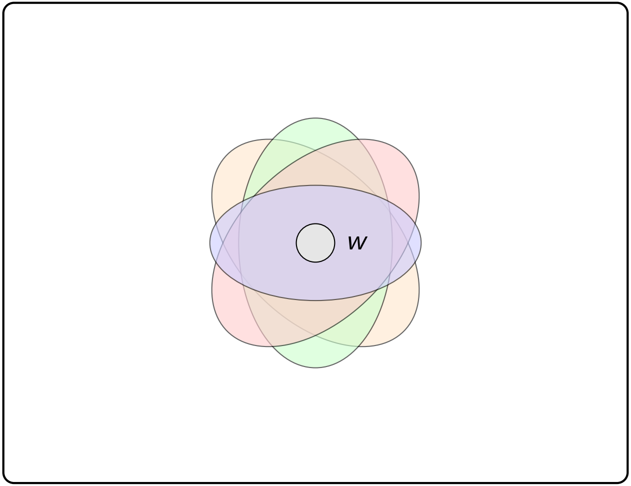
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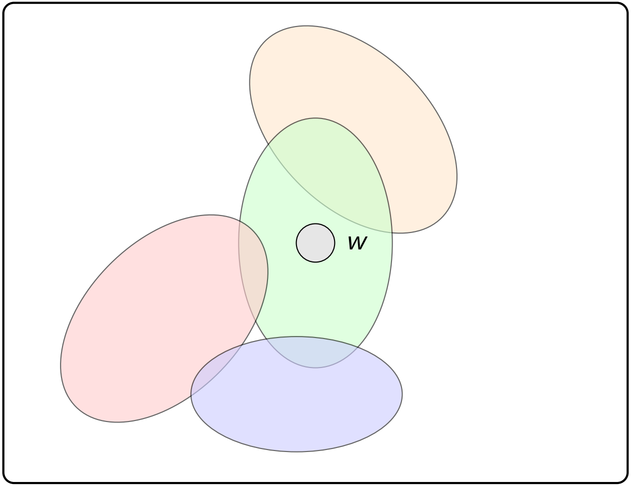
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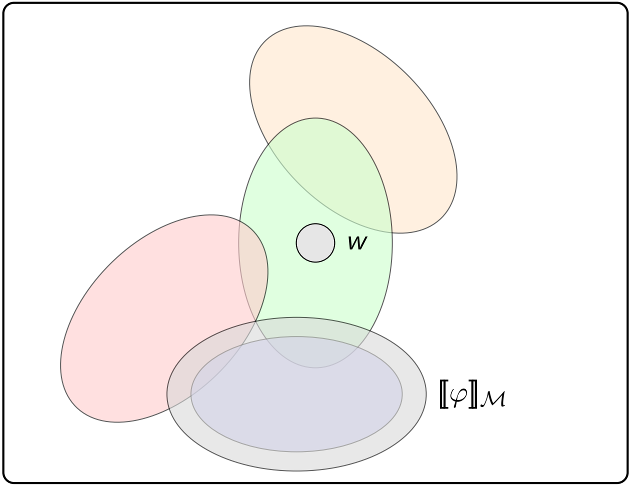


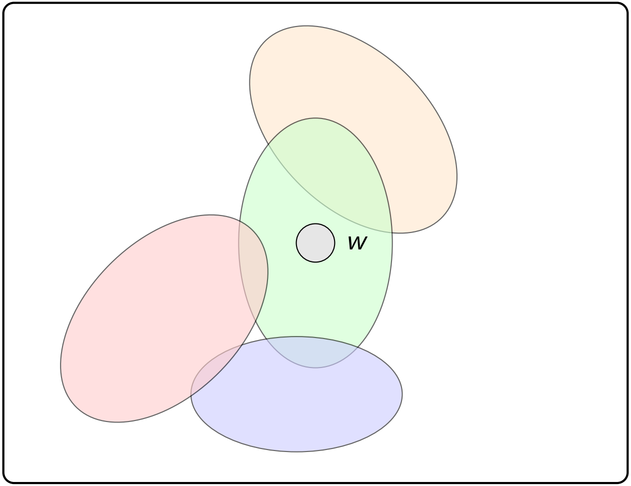
$\mathcal{M}, w \models \Box\varphi$ iff **there is** a neighborhood of w **contained in** $[\![\varphi]\!]_{\mathcal{M}}$

Relational model: $\langle W, R, V \rangle$ where $R : W \to \wp(W)$

$w \models \Box\varphi$ iff $R(w) \subseteq [\![\varphi]\!]$

Neighborhood model: $\langle W, N, V \rangle$ where $N : W \to \wp(\wp(W))$

$w \models \Box\varphi$ iff there is a $X \in N(w)$ such that $X \subseteq [\![\varphi]\!]$



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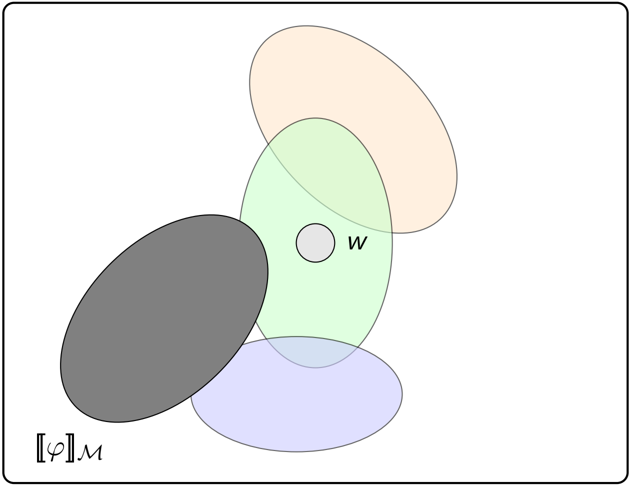

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- ✓ Non-normal modal logics
- ✓ Neighborhood semantics for modal logic

Why non-normal modal logic?

Why neighborhood models?

To see the necessity of the more general approach, we could consider probability operators, conditional necessity, or, to invoke an especially perspicuous example of Dana Scott, the present progressive tense....Thus N might receive the awkward reading 'it is being the case that', in the sense in which 'it is being the case that Jones leaves' is synonymous with 'Jones is leaving'.

(Montague, pg. 73)

R. Montague. *Pragmatics and Intentional Logic*. 1970.

Segerberg's Essay

K. Segerberg. *An Essay on Classical Modal Logic*. Uppsula Technical Report, 1970.

Segerberg's Essay

K. Segerberg. *An Essay on Classical Modal Logic*. Uppsula Technical Report, 1970.

This essay purports to deal with classical modal logic. The qualification "classical" has not yet been given an established meaning in connection with modal logic....Clearly one would like to reserve the label "classical" for a category of modal logics which—if possible—is large enough to contain all or most of the systems which for historical or theoretical reasons have come to be regarded as important, and which also posses a high degree of naturalness and homogeneity.

(pg. 1)

Two routes to a logical framework

1. Identify interesting patterns that you (do not) want to represent
2. Identify interesting structures that you want to reason about

- Logical omniscience
- Logics of knowledge and beliefs
- Logic of high probability
- Logics of ability
- Deontic logics
- Logics of classical deduction
- Logics of group decision making

Logical Omniscience/Knowledge Closure

RM	From $\varphi \to \psi$, infer $\Box\varphi \to \Box\psi$
K	$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$
Nec	From φ, infer $\Box\varphi$
RE	From $\varphi \leftrightarrow \psi$, infer $\Box\varphi \leftrightarrow \Box\psi$

Logical Omniscience/Knowledge Closure

RM From $\varphi \rightarrow \psi$, infer $\Box\varphi \rightarrow \Box\psi$
closure under logical implication

K $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$

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RM From $\varphi \rightarrow \psi$, infer $\Box\varphi \rightarrow \Box\psi$
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K $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$
closure under known implication

Nec From φ, infer $\Box\varphi$

RE From $\varphi \leftrightarrow \psi$, infer $\Box\varphi \leftrightarrow \Box\psi$

Logical Omniscience/Knowledge Closure

RM	From $\varphi \rightarrow \psi$, infer $\Box\varphi \rightarrow \Box\psi$ *closure under logical implication*
K	$\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$ *closure under known implication*
Nec	From φ, infer $\Box\varphi$ *knowledge of all logical validities*
RE	From $\varphi \leftrightarrow \psi$, infer $\Box\varphi \leftrightarrow \Box\psi$

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RM From $\varphi \rightarrow \psi$, infer $\Box\varphi \rightarrow \Box\psi$
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Nec From φ, infer $\Box\varphi$
knowledge of all logical validities

RE From $\varphi \leftrightarrow \psi$, infer $\Box\varphi \leftrightarrow \Box\psi$
closure under logical equivalence

Logical Omniscience/Knowledge Closure

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closure under logical implication

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closure under logical equivalence

Logical Omniscience/Knowledge Closure

W. Holliday. *Epistemic closure and epistemic logic I: Relevant alternatives and subjunctivism*. Journal of Philosophical Logic, 1 - 62, 2014.

J. Halpern and R. Puccella. *Dealing with logical omniscience: Expressiveness and pragmatics*. Artificial Intelligence 175(1), pgs. 220 - 235, 2011.

Logics of High Probability

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H. Kyburg and C.M. Teng. *The Logic of Risky Knowledge*. Proceedings of WoLLIC (2002).

A. Herzig. *Modal Probability, Belief, and Actions*. Fundamenta Informaticae (2003).

R. Stalnaker. *On logics of knowledge and belief*. Philosophical Studies 128, 169 199, 2006.

(K)	$K(\varphi \to \psi) \to (K\varphi \to K\psi)$
(T)	$K\varphi \to \varphi$
(4)	$K\varphi \to KK\varphi$
(Nec)	From φ infer $K\varphi$

(K)	$K(\varphi \to \psi) \to (K\varphi \to K\psi)$
(T)	$K\varphi \to \varphi$
(4)	$K\varphi \to KK\varphi$
(Nec)	From φ infer $K\varphi$

(PI)	$B\varphi \to KB\varphi$
(NI)	$\neg B\varphi \to K\neg B\varphi$
(KB)	$K\varphi \to B\varphi$
(D)	$B\varphi \to \langle B\rangle\varphi$
(SB)	$B\varphi \to BK\varphi$

$$(.2) \qquad \langle K\rangle K\varphi \to K\langle K\rangle\varphi$$

$$(\mathrm{DefKB}) \qquad B\varphi \leftrightarrow \langle K\rangle K\varphi$$

$$(.2) \qquad \langle K\rangle K\varphi \rightarrow K\langle K\rangle\varphi$$

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What happens if we drop axiom (4)?

$$(.2) \qquad \langle K \rangle K\varphi \rightarrow K\langle K \rangle \varphi$$

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Claim. B is a normal modal operator.

What happens if we drop axiom (4)?

Under certain conditions, B is not a normal modal operator.

D. Klein, N. Gratzl, and O. Roy. *Introspection, normality and agglomeration*. Logic, Rationality, and Interaction, 5th Workshop, LORI 2015, 195 206.

Logic of Deduction

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Let $\mathcal{L}_0 \subseteq \mathcal{L}$ be the set of propositional formulas.

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Interpretation: $(\cdot)^* : \mathsf{At} \to \wp(\Sigma)$

- $(\varphi \vee \psi)^* = (\varphi)^* \cup (\psi)^*$
- $(\neg\varphi)^* = \Sigma - (\varphi)^*$
- $(\Box\varphi)^* = \{\alpha \in \Sigma \mid (\varphi)^* \vdash \alpha\}$

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P. Naumov. *On modal logic of deductive closure*. APAL (2005).

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Fact: $\Box\varphi \wedge \Box\psi \to \Box(\varphi \wedge \psi)$ is not valid.

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Validities: $\varphi \rightarrow \Box\varphi$,

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P. Naumov. *On modal logic of deductive closure*. APAL (2005).

(M)	$\Box(\varphi \wedge \psi) \rightarrow \Box\varphi \wedge \Box\psi$
(C)	$\Box\varphi \wedge \Box\psi \rightarrow \Box(\varphi \wedge \psi)$
(N)	$\Box\top$
(K)	$\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$
(Dual)	$\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$
(Nec)	from $\vdash \varphi$ infer $\vdash \Box\varphi$
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(M) $\Box(\varphi \wedge \psi) \rightarrow \Box\varphi \wedge \Box\psi$

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Deontic Logic

$\Box\varphi$ mean "*it is obliged that* φ."

$$\frac{\varphi \to \psi}{\Box\varphi \to \Box\psi}$$

J. Forrester. *Paradox of Gentle Murder*. 1984.

L. Goble. *Murder Most Gentle: The Paradox Deepens*. 1991.

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1. Jones murders Smith
2. Jones ought not to murder Smith

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Deontic Logic

$\Box\varphi$ mean "*it is obliged that* φ."

1. Jones murders Smith
2. Jones ought not to murder Smith
3. If Jones murders Smith, then Jones ought to murder Smith gently

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$\Rightarrow$ If Jones murders Smith gently, then Jones murders Smith.

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(Mon) If Jones ought to murder Smith gently, then Jones ought to murder Smith

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✓ Jones ought to murder Smith gently

5. If Jones murders Smith gently, then Jones murders Smith.

✓ If Jones ought to murder Smith gently, then Jones ought to murder Smith

7. Jones ought to murder Smith

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✗ Jones ought not to murder Smith
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4. Jones ought to murder Smith gently
5. If Jones murders Smith gently, then Jones murders Smith.
6. If Jones ought to murder Smith gently, then Jones ought to murder Smith
✗ Jones ought to murder Smith

J. Forrester. *Paradox of Gentle Murder*. 1984.

L. Goble. *Murder Most Gentle: The Paradox Deepens*. 1991.

Why non-normal modal logic? ✓

Why neighborhood models?

- Subset spaces, neighborhood frames/models, reasoning about subset spaces
- Interesting mathematical structures: Ultrafilters, topologies, hypergraphs
- Logic of knowledge, evidence and belief
- Coalitional logic

Some Terminology: Subset Spaces

Let W be a set and $\mathcal{F} \subseteq \wp(W)$.

- $\mathcal{F}$ is closed under intersections if for any collections of sets $\{X_i\}_{i \in I}$ such that for each $i \in I$, $X_i \in \mathcal{F}$, then $\cap_{i \in I} X_i \in \mathcal{F}$.
- $\mathcal{F}$ is closed under unions if for any collections of sets $\{X_i\}_{i \in I}$ such that for each $i \in I$, $X_i \in \mathcal{F}$, then $\cup_{i \in I} X_i \in \mathcal{F}$.
- $\mathcal{F}$ is closed under complements if for each $X \subseteq W$, if $X \in \mathcal{F}$, then $X^C \in \mathcal{F}$.
- $\mathcal{F}$ is supplemented, or closed under supersets or monotonic provided for each $X \subseteq W$, if $X \in \mathcal{F}$ and $X \subseteq Y \subseteq W$, then $Y \in \mathcal{F}$.

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Let W be a set and $\mathcal{F} \subseteq \wp(W)$.

- $\mathcal{F}$ contains the unit provided $W \in \mathcal{F}$
- the set $\cap_{X \in \mathcal{F}} X$ the core of $\mathcal{F}$. $\mathcal{F}$ contains its core provided $\cap_{X \in \mathcal{F}} X \in \mathcal{F}$.
- $\mathcal{F}$ is proper if $X \in \mathcal{F}$ implies $X^C \notin \mathcal{F}$.
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- the set $\cap_{X \in \mathcal{F}} X$ the core of $\mathcal{F}$. $\mathcal{F}$ contains its core provided $\cap_{X \in \mathcal{F}} X \in \mathcal{F}$.
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- $\mathcal{F}$ is consistent if $\emptyset \notin \mathcal{F}$
- $\mathcal{F}$ is normal if $\mathcal{F} \neq \emptyset$.

A few more definitions

- $\mathcal{F}$ is a filter if $\mathcal{F}$ contains the unit, closed under binary intersections and supplemented. $\mathcal{F}$ is a proper filter if in addition $\mathcal{F}$ does not contain the emptyset.
- $\mathcal{F}$ is an ultrafilter if $\mathcal{F}$ is proper filter and for each $X \subseteq W$, either $X \in \mathcal{F}$ or $X^C \in \mathcal{F}$.
- $\mathcal{F}$ is a topology if $\mathcal{F}$ contains the unit, the emptyset, is closed under finite intersections and arbitrary unions.
- $\mathcal{F}$ is augmented if $\mathcal{F}$ contains its core and is supplemented.

Neighborhood Frames

Let W be a non-empty set of states.

Any function $N : W \to \wp(\wp(W))$ is called a neighborhood function

A pair $\langle W, N\rangle$ is a called a neighborhood frame if W a non-empty set and N is a neighborhood function.

A neighborhood model based on $\mathfrak{F} = \langle W, N\rangle$ is a tuple $\langle W, N, V\rangle$ where $V : \mathrm{At} \to \wp(W)$ is a valuation function.

Truth in a Model

- $\mathfrak{M}, w \models p$ iff $w \in V(p)$
- $\mathfrak{M}, w \models \neg\varphi$ iff $\mathfrak{M}, w \not\models \varphi$
- $\mathfrak{M}, w \models \varphi \wedge \psi$ iff $\mathfrak{M}, w \models \varphi$ and $\mathfrak{M}, w \models \psi$

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- $\mathfrak{M}, w \models \Box\varphi$ iff $[\![\varphi]\!]_{\mathfrak{M}} \in N(w)$
- $\mathfrak{M}, w \models \Diamond\varphi$ iff $W - [\![\varphi]\!]_{\mathfrak{M}} \not\in N(w)$

where $[\![\varphi]\!]_{\mathfrak{M}} = \{w \mid \mathfrak{M}, w \models \varphi\}$.

Let $N : W \to \wp\wp W$ be a neighborhood function and define $m_N : \wp W \to \wp W$:

$$\text{for } X \subseteq W,\ m_N(X) = \{w \mid X \in N(w)\}$$

1. $[\![p]\!]_{\mathfrak{M}} = V(p)$ for $p \in \mathsf{At}$
2. $[\![\neg\varphi]\!]_{\mathfrak{M}} = W - [\![\varphi]\!]_{\mathfrak{M}}$
3. $[\![\varphi \wedge \psi]\!]_{\mathfrak{M}} = [\![\varphi]\!]_{\mathfrak{M}} \cap [\![\psi]\!]_{\mathfrak{M}}$
4. $[\![\Box\varphi]\!]_{\mathfrak{M}} = m_N([\![\varphi]\!]_{\mathfrak{M}})$
5. $[\![\Diamond\varphi]\!]_{\mathfrak{M}} = W - m_N(W - [\![\varphi]\!]_{\mathfrak{M}})$

Detailed Example

Suppose $W = \{w, s, v\}$ is the set of states and define a neighborhood model $\mathfrak{M} = \langle W, N, V \rangle$ as follows:

- $N(w) = \{\{s\}, \{v\}, \{w, v\}\}$
- $N(s) = \{\{w, v\}, \{w\}, \{w, s\}\}$
- $N(v) = \{\{s, v\}, \{w\}, \emptyset\}$

Further suppose that $V(p) = \{w, s\}$ and $V(q) = \{s, v\}$.

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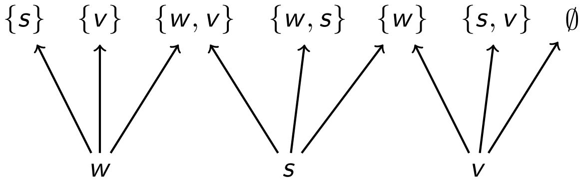

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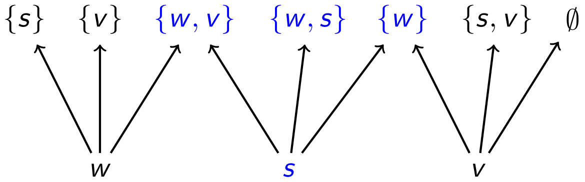

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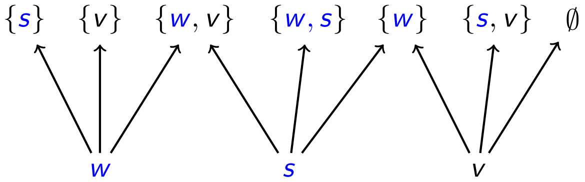

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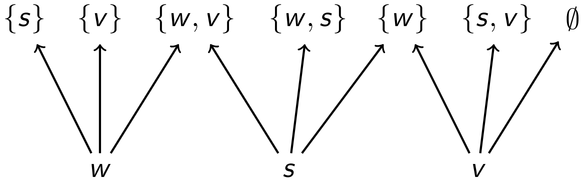

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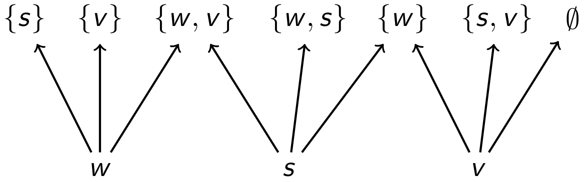

$$\mathfrak{M}, s \models \Box p$$

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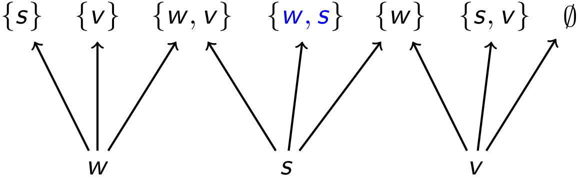

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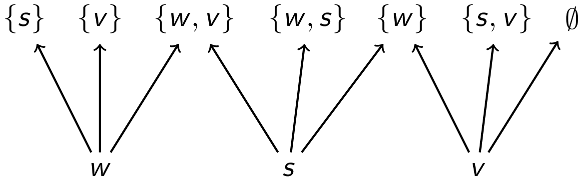

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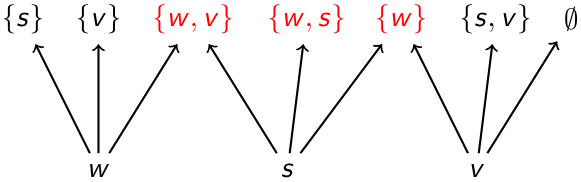

$$\mathfrak{M}, s \models \Diamond p$$

$$[\![\neg p]\!]_{\mathfrak{M}} = \{v\}$$

Detailed Example

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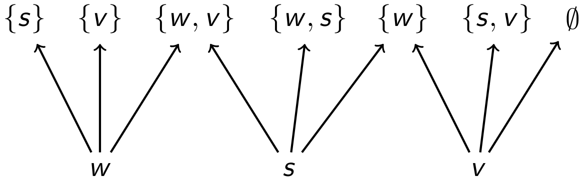

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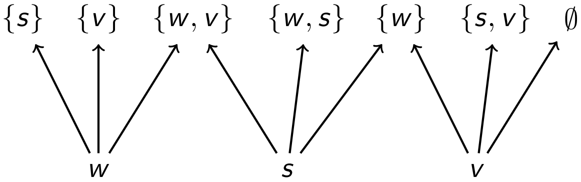

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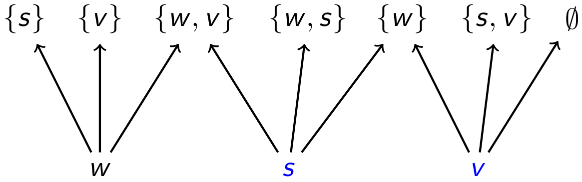

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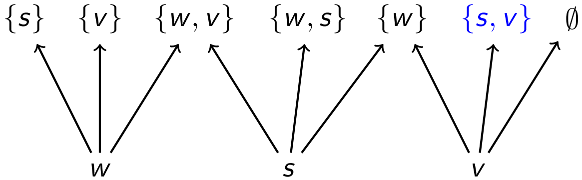

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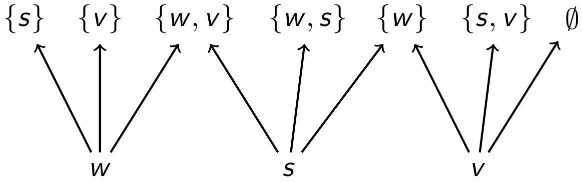

$\mathfrak{M}, w \not\models \Diamond\Box p$ $\qquad$ $\mathfrak{M}, v \models \Box\Diamond p$

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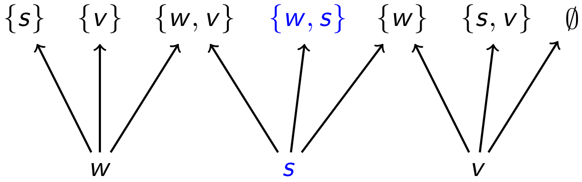

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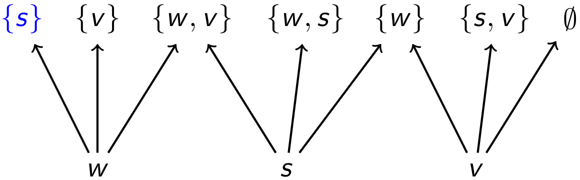

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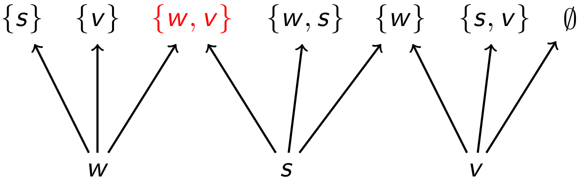

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Other modal operators

- $\mathfrak{M}, w \models \langle\ \rangle\varphi$ iff $\exists X \in N(w)$ such that $\exists v \in X$, $\mathfrak{M}, v \models \varphi$
- $\mathfrak{M}, w \models [\ ]\varphi$ iff $\forall X \in N(w)$ such that $\forall v \in X$, $\mathfrak{M}, v \models \varphi$

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Lemma

Let $\mathfrak{M} = \langle W, N, V \rangle$ be a neighborhood model. The for each $w \in W$,

1. *if $\mathfrak{M}, w \models \Box\varphi$ then $\mathfrak{M}, w \models \langle\ ]\varphi$*
2. *if $\mathfrak{M}, w \models [\ \rangle\varphi$ then $\mathfrak{M}, w \models \Diamond\varphi$*

However, the converses of the above statements are false.

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Lemma

1. *If $\varphi \to \psi$ is valid in $\mathfrak{M}$, then so is $\langle\ ]\varphi \to \langle\ ]\psi$.*
2. $\langle\ ](\varphi \wedge \psi) \to (\langle\ ]\varphi \wedge \langle\ ]\psi)$ *is valid in $\mathfrak{M}$*

Investigate analogous results for the other modal operators defined above.

Digression: Subset Space Models

L. Moss and R. Parikh. *Topological Reasoning and The Logic of Knowledge*. TARK (1992).

Subset Models

A **Subset Frame** is a pair $\langle W, \mathcal{O} \rangle$ where

- W is a set of states
- $\mathcal{O} \subseteq \wp(W)$ is a set of subsets of W, i.e., a set of *observations*

Neighborhood Situation: Given a subset frame $\langle W, \mathcal{O} \rangle$, (w, U) is called a neighborhood situation, provided $w \in U$ and $U \in \mathcal{O}$.

Model: $\langle W, \mathcal{O}, V \rangle$, where V : At $\rightarrow \wp(W)$ is a valuation function.

Language: $\varphi := p \mid \varphi \wedge \varphi \mid \neg\varphi \mid K\varphi \mid \Diamond\varphi$.

Truth in a subset model

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- $w, U \models K\varphi$ iff for all $v \in U$, $v, U \models \varphi$
- $w, U \models \Diamond\varphi$ iff there is a $V \in \mathcal{O}$ such that $w \in V \subseteq U$ and $w, V \models \varphi$

Axioms

1. All propositional tautologies
2. $(p \to \Box p) \wedge (\neg p \to \Box \neg p)$, for $p \in \mathsf{At}$.
3. $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$
4. $\Box\varphi \to \varphi$
5. $\Box\varphi \to \Box\Box\varphi$
6. $K(\varphi \to \psi) \to (K\varphi \to K\psi)$
7. $K\varphi \to \varphi$
8. $K\varphi \to KK\varphi$
9. $\neg K\varphi \to K\neg K\varphi$
10. $K\Box\varphi \to \Box K\varphi$

We include the following rules: modus ponens, K_i-necessitation and $\Box$-necessitation.

Theorem

The previous axioms are sound and complete for the class of all subset models.

L. Moss and R. Parikh. *Topological Reasoning and The Logic of Knowledge*. TARK (1992).

Fact: $\Box\Diamond\varphi \rightarrow \Diamond\Box\varphi$ is sound for spaces closed under intersections.

Fact: $\Diamond\varphi \wedge L\Diamond\psi \rightarrow \Diamond[\Diamond\varphi \wedge L\Diamond\psi \wedge K\Diamond L(\varphi \vee \psi)]$ is sound for spaces closed under binary unions.

Overview of Results

- (Georgatos: 1993, 1994, 1997) completely axiomatized Topologic where $\mathcal{O}$ is restricted to a topology and showed that the logic has the finite model property. Similarly for treelike spaces.
- (Weiss and Parikh: 2002) showed that an infinite number of axiom schemes is required to axiomatize Topologics in which $\mathcal{O}$ is closed under intersection.
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