Epistemic Game Theory

Lecture 3

ESSLLI'12, Opole

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August 8, 2012

Subgames

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A **restriction** of H is a sequence $G = (G_1, \ldots, G_n)$ such that $G_i \subseteq H_i$ for all $i \in \{1, \ldots, n\}$.

The set of all restrictions of a game H ordered by componentwise set inclusion forms a complete lattice.

Game Models

Relational models: $\langle W, R_i \rangle$ where $R_i \subseteq W \times W$. Write $R_i(w) = \{v \mid wR_iv\}$.

Events: $E \subseteq W$

Knowledge/Belief: $\Box E = \{w \mid R_i(w) \subseteq E\}$

Common knowledge/belief:
$\Box^1 E = \Box E$
$\Box^{k+1} E = \Box\Box^k E$
$\Box^* E = \bigcap_{k=1}^{\infty} \Box^k E$

Fact. An event F is called **evident** provided $F \subseteq \Box F$. $w \in \Box^* E$ provided there is an evident event F such that $w \in F \subseteq \Box E$.

Game Models

Let $G = (G_1, \ldots, G_n)$ be a restriction of a game H.

A **knowledge/belief model of** G is a tuple $\langle W, R_1, \ldots, R_n, \sigma_1, \ldots, \sigma_n \rangle$ where $\langle W, R_1, \ldots, R_n \rangle$ is a knowledge/belief model and $\sigma_i : W \to G_i$.

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Given a model $\langle W, R_1, \ldots, R_n, \sigma_1, \ldots \sigma_n \rangle$ for a restriction G and a sequence $\overline{E} = \{E_1, \ldots, E_n\}$ where $E_i \subseteq W$,

$$G_{\overline{E}} = (\sigma_1(E_1), \ldots, \sigma_n(E_n))$$

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- T^∞ is the "outcome of T: $T^0 = \top$, $T^{\alpha+1} = T(T^\alpha)$, $T^\beta = \bigcap_{\alpha<\beta} T^\alpha$, The outcome of iterating T is the least α such that $T^{\alpha+1} = T^\alpha$, denoted T^∞

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- **Tarski's Fixed-Point Theorem**: Every monotonic operator T has a (least and largest) fixed point $T^\infty = \nu T = \bigcup\{G \mid G \subseteq T(G)\}$.
- T is contracting if $T(G) \subseteq G$. Every contracting operator has an outcome (T^∞ is well-defined)

Rationality Properties

$\varphi(s_i, G_i, G_{-i})$ holds between a strategy $s_i \in H_i$, a set of strategies G_i for player i and strategies G_{-i} of the opponents. Intuitively s_i is φ-optimal strategy for player i in the restricted game $\langle G_i, G_{-i}, u_1, \ldots, u_n \rangle$ (where the payoffs are suitably restricted).

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φ_i is **monotonic** if for all G_{-i}, $G'_{-i} \subseteq H_{-i}$ and $s_i \in H_i$

$$G_{-i} \subseteq G'_{-i} \text{ and } \varphi(s_i, H_i, G_{-i}) \text{ implies } \varphi(s_i, H_i, G'_{-i})$$

Removing Strategies

If $\varphi = (\varphi_1, \ldots, \varphi_n)$, then define $T_\varphi(G) = G'$ where

- $G = (G_1, \ldots, G_n)$, $G' = (G'_1, \ldots, G'_n)$,
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T_φ is contracting, so it has an outcome T_φ^∞

If each φ_i is monotonic, then νT_φ exists and equals T_φ^∞.

Rational Play

Let $H = \langle H_1, \ldots, H_n, u_1, \ldots, u_n \rangle$ a strategic game and $\langle W, R_1, \ldots, R_n, \sigma_1, \ldots, \sigma_n \rangle$ a model for H.

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$\mathbf{Rat}(\varphi) = \{w \in W \mid \text{ each player is } \varphi_i\text{-rational in } w\}$

$\Box\mathbf{Rat}(\varphi)$
$\Box^*\mathbf{Rat}(\varphi)$

Theorem (Apt and Zvesper).

- Suppose that each φ_i is monotonic. Then for all belief models for H,

$$G_{\mathbf{Rat}(\varphi) \cap B^*(\mathbf{Rat}(\varphi))} \subseteq T_\varphi^\infty$$

- Suppose that each φ_i is monotonic. Then for all knowledge models for H,

$$G_{K^*(\mathbf{Rat}(\varphi))} \subseteq T_\varphi^\infty$$

- For some standard knowledge model for H,

$$T_\varphi^\infty \subseteq G_{K^*(\mathbf{Rat}(\varphi))}$$

K. Apt and J. Zvesper. *The Role of Monotonicity in the Epistemic Analysis of Games*. Games, 1(4), pgs. 381-394, 2010.

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there is an F such that $F \subseteq \Box F$ and

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Since each φ_i is monotonic, T_φ is monotonic and by Tarski's fixed-point theorem, $G_{F\cap\mathbf{Rat}(\varphi)} \subseteq T_\varphi^\infty$. But $s_i = \sigma_i(w)$ and $w \in F \cap \mathbf{Rat}(\varphi)$, so s_i is the ith component in T_φ^∞.

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Since $w' \in \mathbf{Rat}(\varphi)$, $\varphi_i(\sigma_i(w'), H_i, (G_{R_i(w)})_{-i})$ holds.

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This implies $(G_{R_i(w')}) \subseteq (G_{F \cap \mathbf{Rat}(\varphi)})_{-i}$, and so by monotonicity of φ_i, $\varphi_i(s_i, H_i, (G_{F \cap \mathbf{Rat}(\varphi)})_{-i})$ holds.

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This means $G_{F \cap \mathbf{Rat}(\varphi)} \subseteq T_\varphi(G_{F \cap \mathbf{Rat}(\varphi)})$

$sd_i(s_i, G_i, G_{-i})$ is $\neg\exists s_i' \in G_i, \forall s_{-i} \in G_{-i} u_i(s_i', s_{-i}) > u_i(s_i, s_{-i})$

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$br_i(s_i, G_i, G_{-i})$ is $\exists \mu_i \in \mathcal{B}_i(G_{-i}) \forall s_i' \in G_i, U_i(s_i, \mu_i) \geq U_i(s_i', \mu_i)$.

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$U_\varphi(G) = G'$ where $G_i' = \{s_i \in G_i \mid \varphi_i(s_i, G_i, G_{-i})\}$.

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Note: U_φ is *not* monotonic.

Corollary. For all belief models, $G_{\mathbf{Rat}(br)\cap\Box^*\mathbf{Rat}(br)} \subseteq U^\infty_{sd}$. For all G, we have

$$T_{br}(G) \subseteq T_{sd}(G)$$

$$T_{sd}(G) \subseteq U_{sd}(G)$$

Then, $T^\infty_{sd} \subseteq U^\infty_{sd}$.

Corollary. For all belief models, $G_{\mathbf{Rat}(br)\cap\Box^*\mathbf{Rat}(br)} \subseteq U_{sd}^{\infty}$. For all G, we have

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Then, $T_{sd}^{\infty} \subseteq U_{sd}^{\infty}$.

Fact. Consider two operators T_1, T_2 on $(D, \subseteq)$ such that,

- for all G, $T_1(G) \subseteq T_2(G)$
- T_1 is monotonic
- T_2 is contracting

Then, $T_1^{\infty} \subseteq T_2^{\infty}$.

This analysis does not work for weak dominance...