Epistemic Game Theory

Lecture 2

ESSLLI'12, Opole

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Plan for the week

1. **Monday** Basic Concepts.
2. **Tuesday** Epistemics.
   - Relating dominance reasoning with maximizing expected utility
   - Probabilistic/graded models of beliefs, knowledge and higher-order attitudes.
   - Logical/qualitative models of beliefs, knowledge and higher-order attitudes.
3. **Wednesday** Fundamentals of Epistemic Game Theory.
4. **Thursday** Puzzles and Paradoxes.
5. **Friday** Extensions and New Directions.

Dominance vs MEU

		Bob	
		L	*R*
Ann	*U*	3,3	0,0
	D	0,0	1,1

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- **Ann's beliefs**: $p_A \in \Delta(\{L, R\})$ with $p_A(L) = 1/6$
  Bob's beliefs: $p_B \in \Delta(\{U, D\})$ with $p_B(U) = 3/4$.

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  $EU(U, p_A) = p_A(L) \cdot u_A(U, L) + p_A(R) \cdot u_A(U, R)$
  $EU(D, p_A) = p_A(L) \cdot u_A(D, L) + p_A(R) \cdot u_A(D, R)$

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  $$EU(D, p_A) = 1/6 \cdot 0 + 5/6 \cdot 1 = 0.833$$
  $$EU(L, p_B) = p_B(U) \cdot u_B(U, L) + p_B(D) \cdot u_B(D, R)$$
  $$EU(R, p_B) = p_B(U) \cdot u_B(U, R) + p_B(D) \cdot u_B(D, R)$$

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Comparing Dominance Reasoning and MEU

$G = \langle N, \{S_i\}_{i \in N}, \{u_i\}_{i \in N} \rangle$

$X \subseteq S_{-i}$ (a set of strategy profiles for all players except i)

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$$\forall s_{-i} \in X, \quad u_i(s, s_{-i}) > u_i(s', s_{-i})$$

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$s, s' \in S_i$, s **weakly dominates** s' with respect to X provided

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$p \in \Delta(X)$, s is a **best response** to p with respect to X provided

$$\forall s' \in S_i, \quad EU(s, p) \geq EU(s', p)$$

Strict Dominance and MEU

Fact. Suppose that $G = \langle N, \{S_i\}_{i\in N}, \{u_i\}_{i\in N}\rangle$ is a strategic game and $X \subseteq S_{-i}$. A strategy $s_i \in S_i$ is strictly dominated (possibly by a mixed strategy) with respect to X iff there is no probability measure $p \in \Delta(X)$ such that s_i is a best response to p.

Dominance vs MEU

Suppose that $G = \langle N, \{S_i\}_{i\in N}, \{u_i\}_{i\in N}\rangle$ is a finite strategic game.

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$$\exists s_i' \in S_i, \forall s_{-i} \in X, \quad u_i(s_i', s_{-i}) > u_i(s_i, s_{-i})$$

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Let $p \in \Delta(X)$ be any probability measure. Then,

$$\forall s_{-i} \in X, \quad p(s_{-i}) \cdot u_i(s_i', s_{-i}) \geq p(s_{-i}) \cdot u_i(s_i, s_{-i})$$

$$\exists s_{-i} \in X, \quad p(s_{-i}) \cdot u_i(s_i', s_{-i}) > p(s_{-i}) \cdot u_i(s_i, s_{-i})$$

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Hence,

$$\sum_{s_{-i} \in S_{-i}} p(s_{-i}) \cdot u_i(s_i', s_{-i}) > \sum_{s_{-i} \in S_{-i}} p(s_{-i}) \cdot u_i(s_i, s_{-i})$$

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Hence,

$$\sum_{s_{-i} \in S_{-i}} p(s_{-i}) \cdot u_i(s_i', s_{-i}) > \sum_{s_{-i} \in S_{-i}} p(s_{-i}) \cdot u_i(s_i, s_{-i})$$

So, $EU(s_i', p) > EU(s_i, p)$: s_i is not a best response to p.

For the converse direction, we sketch the proof for two player games and where $X = S_{-i}$. [1]

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Let $G = \langle S_1, S_2, u_1, u_2 \rangle$ be a two-player game.
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Suppose that $\alpha \in \Delta(S_1)$ is not a best response to any $p \in \Delta(S_2)$.

$$\forall p \in \Delta(S_2) \;\; \exists q \in \Delta(S_1), \quad U_1(q, p) > U_1(\alpha, p)$$

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Suppose that $\alpha \in \Delta(S_1)$ is not a best response to any $p \in \Delta(S_2)$.

$$\forall p \in \Delta(S_2) \;\; \exists q \in \Delta(S_1), \quad U_1(q, p) > U_1(\alpha, p)$$

We can define a function $b : \Delta(S_2) \rightarrow \Delta(S_1)$ where, for each $p \in \Delta(S_2)$, $U_1(b(p), p) > U_1(\alpha, p)$.

---

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Consider the game $G' = \langle S_1, S_2, \overline{u}_1, \overline{u}_2 \rangle$ where

$$\overline{u}_1(s_1, s_2) = u_1(s_1, s_2) - U_1(\alpha, s_2) \text{ and } \overline{u}_2(s_1, s_2) = -\overline{u}_1(s_1, s_2)$$

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By the minimax theorem, there is a Nash equilibrium (p_1^*, p_2^*) such that for all $m \in \Delta(S_2)$,

$$\overline{U}(p_1^*, m) \geq \overline{U}_1(p_1^*, p_2^*) \geq \overline{U}_1(b(p_2^*), p_2^*)$$

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We now prove that $\overline{U}_1(b(p_2^*), p_2^*) > 0$:

Dominance vs MEU

$$\overline{U}_1(b(p_2^*), p_2^*) \quad = \quad \sum_{x \in S_1} \sum_{y \in S_2} b(p_2^*)(x) p_2^*(y) \overline{u}_1(x, y)$$

Dominance vs MEU

$$\begin{aligned}\overline{U}_1(b(p_2^*), p_2^*) &= \textstyle\sum_{x\in S_1}\sum_{y\in S_2} b(p_2^*)(x)p_2^*(y)\overline{u}_1(x,y)\\ &= \textstyle\sum_{x\in S_1}\sum_{y\in S_2} b(p_2^*)(x)p_2^*(y)[u_1(x,y) - U_1(\alpha, y)]\end{aligned}$$

Dominance vs MEU

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&= \textstyle\sum_{x\in S_1}\sum_{y\in S_2} b(p_2^*)(x)p_2^*(y)[u_1(x,y) - U_1(\alpha,y)]\\
&= \textstyle\sum_{x\in S_1}\sum_{y\in S_2} b(p_2^*)(x)p_2^*(y)u_1(x,y)\\
&- \textstyle\sum_{x\in S_1}\sum_{y\in S_2} b(p_2^*)(x)p_2^*(y)U_1(\alpha,y)
\end{aligned}$$

Dominance vs MEU

$$
\begin{array}{rcl}
\overline{U}_1(b(p_2^*), p_2^*) & = & \sum_{x\in S_1}\sum_{y\in S_2} b(p_2^*)(x)p_2^*(y)\overline{u}_1(x,y) \\[2ex]
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 & = & \sum_{x\in S_1}\sum_{y\in S_2} b(p_2^*)(x)p_2^*(y)u_1(x,y) \\
 & - & \sum_{x\in S_1}\sum_{y\in S_2} b(p_2^*)(x)p_2^*(y)U_1(\alpha,y) \\[2ex]
 & = & U_1(b(p_2^*), p_2^*) \\
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& = & \sum_{x \in S_1} \sum_{y \in S_2} b(p_2^*)(x) p_2^*(y) u_1(x, y) \\
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& = & U_1(b(p_2^*), p_2^*) \\
& - & \sum_{x \in S_1} \sum_{y \in S_2} b(p_2^*)(x) p_2^*(y) U_1(\alpha, y) \\
& > & U_1(\alpha, p_2^*) - \sum_{x \in S_1} \sum_{y \in S_2} b(p_2^*)(x) p_2^*(y) U_1(\alpha, y)
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& = & U_1(\alpha, p_2^*) - U_1(\alpha, p_2^*)\cdot\sum_{x\in S_1} b(p_2^*)(x)
\end{array}$$

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\overline{U}_1(b(p_2^*), p_2^*) & = & \sum_{x \in S_1} \sum_{y \in S_2} b(p_2^*)(x) p_2^*(y) \overline{u}_1(x,y) \\
& = & \sum_{x \in S_1} \sum_{y \in S_2} b(p_2^*)(x) p_2^*(y) [u_1(x,y) - U_1(\alpha, y)] \\
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& = & U_1(\alpha, p_2^*) - U_1(\alpha, p_2^*) \cdot \sum_{x \in S_1} b(p_2^*)(x) \\
& = & U_1(\alpha, p_2^*) - U_1(\alpha, p_2^*) = 0
\end{array}$$

Dominance vs MEU

Hence, for all $m \in \Delta(S_2)$ we have

$$\overline{U}(p_1^*, m) \geq \overline{U}_1(p_1^*, p_2^*) \geq \overline{U}_1(b(p_2^*), p_2^*) > 0$$

Hence, for all $m \in \Delta(S_2)$ we have

$$\overline{U}(p_1^*, m) \geq \overline{U}_1(p_1^*, p_2^*) \geq \overline{U}_1(b(p_2^*), p_2^*) > 0$$

which implies for all $m \in \Delta(S_2)$, $U_1(p_1^*, m) > U_1(\alpha, m)$, and so α is strictly dominated by p_1^*.

Important Issue: Correlated Beliefs

x	l	r
u	1,1,3	1,0,3
d	0,1,0	0,0,0

y	l	r
u	1,1,2	1,0,0
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z	l	r
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- Note that y is not strictly dominated for Charles.

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- Note that y is not strictly dominated for Charles.
- It is easy to find a probability measure $p \in \Delta(S_A \times S_B)$ such that y is a best response to p. Suppose that $p(u,l) = p(d,r) = \frac{1}{2}$. Then, $EU(x,p) = EU(z,p) = 1.5$ while $EU(y,p) = 2$.

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- Note that y is not strictly dominated for Charles.
- It is easy to find a probability measure $p \in \Delta(S_A \times S_B)$ such that y is a best response to p. Suppose that $p(u, l) = p(d, r) = \frac{1}{2}$. Then, $EU(x, p) = EU(z, p) = 1.5$ while $EU(y, p) = 2$.
- However, there is no probability measure $p \in \Delta(S_A \times S_B)$ such that y is a best response to p and $p(u, l) = p(u) \cdot p(l)$.

Dominance vs MEU

x	l	r
u	1,1,3	1,0,3
d	0,1,0	0,0,0

y	l	r
u	1,1,2	1,0,0
d	0,1,0	1,1,2

z	l	r
u	1,1,0	1,0,0
d	0,1,3	0,0,3

- To see this, suppose that a is the probability assigned to u and b is the probability assigned to l. Then, we have:
  - The expected utility of y is $2ab + 2(1-a)(1-b)$;
  - The expected utility of x is $3ab + 3a(1-b) = 3a(b + (1-b)) = 3a$; and
  - The expected utility of z is $3(1-a)b + 3(1-a)(1-b) = 3(1-a)(b + (1-b)) = 3(1-a)$.

Weak Dominance and MEU

Fact. Suppose that $G = \langle N, \{S_i\}_{i\in N}, \{u_i\}_{i\in N}\rangle$ is a strategic game and $X \subseteq S_{-i}$. A strategy $s_i \in S_i$ is weakly dominated (possibly by a mixed strategy) with respect to X iff there is **no full support probability measure** $p \in \Delta^{>0}(X)$ such that s_i is a best response to p.

Dominance vs MEU

Some preliminary remarks

Propositional Attitudes

- We will talk about so-called **propositional attitudes**. These are attitudes (like knowledge, beliefs, desires, intentions, etc...) that take propositions as objects.

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- Proposition will be taken to be element of a given algebra. I.e. measurable subsets of a state space (sigma- and/or power-set algebra), formulas in a given language (abstract Boolean algebra)...

All-out vs graded attitudes

- A propositional attitude A is **all-out** when, for any proposition p, the agent can only be in **three states** of that attitude regarding p:
    1. Ap: the agent "believes" that p.
    2. $A\neg p$: the agent "disbelieve" that p.
    3. $\neg Ap \wedge \neg A\neg p$: the agent "suspends judgment" about p.

All-out vs graded attitudes

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  3. $\neg Ap \wedge \neg A\neg p$: the agent “suspends judgment” about p.

- A propositional attitude A is **graded** when, for any proposition p, the states of that attitude that the agent be in w.r.t. a proposition p can be compared according to their **strength** on a given scale.

p_i	P	$\neg P$
A	1/8	3/8

Hard and Soft Attitudes

- Hard attitudes:
  - Truthful.
  - Unrevisable.
  - Fully introspective.
- Soft attitudes:
  - Can be false / mistaken.
  - Revisable / can be reversed.
  - Not fully introspective.

Models of graded beliefs

Harsanyi Type Space

Based on the work of John Harsanyi on games with *incomplete information*, game theorists have developed an elegant formalism that makes precise talk about beliefs, knowledge and rationality:

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- Each **type** is assigned a joint probability over the space of types and actions

$$\lambda_i : T_i \rightarrow \Delta(T_{-i} \times S_{-i})$$

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Player i's types

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The set of all probability distributions

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The other players' types

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The other players' choices

Returning to the Example: A Game Model

		Bob	
		H	M
Ann	H	3,3	0,0
	M	0,0	1,1

	H	M
t_B	0	0.5
u_B	0.2	0.3

t_A

	H	M
t_A	0	1

t_B

	H	M
t_A	0.4	0.6

u_B

Returning to the Example: A Game Model

Bob

Ann		H	M
	H	3,3	0,0
	M	0,0	1,1

- One type for Ann (t_A) and two types for Bob (t_B, u_B)

	H	M
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- A **state** is a tuple of choices and types: (M, M, t_A, t_B)

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t_B	0	0.5
u_B	0.2	0.3

t_A

	H	M
t_A	0	1

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- One type for Ann (t_A) and two types for Bob (t_B, u_B)
- A **state** is a tuple of choices and types: (M, t_A, M, u_B)
- Calculate **expected utility** in the usual way...

	H	M
t_B	0	0.5
u_B	0.2	0.3

t_A

	H	M
t_A	0	1

t_B

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Returning to the Example: A Game Model

Bob

Ann	H	M
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M	0,0	1,1

- M is **rational** for Ann (t_A)
  $0 \cdot 0.2 + 1 \cdot 0.8 \geq 3 \cdot 0.2 + 0 \cdot 0.8$

	H	M
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t_A

	H	M
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  $0 \cdot 0 + 1 \cdot 1 \geq 3 \cdot 0 + 0 \cdot 1$

	H	M
t_B	0	0.5
u_B	0.2	0.3

t_A

	H	M
t_A	0	1

t_B

	H	M
t_A	0.4	0.6

u_B

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- M is **rational** for Bob (t_B)
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- Ann thinks Bob may be irrational

	H	M
t_B	0	0.5
u_B	0.2	0.3

t_A

	H	M
t_A	0	1

t_B

	H	M
t_A	0.4	0.6

u_B

Returning to the Example: A Game Model

		Bob	
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	M	0,0	1,1

- M is **rational** for Ann (t_A)
  $0 \cdot 0.2 + 1 \cdot 0.8 \geq 3 \cdot 0.2 + 0 \cdot 0.8$
- M is **rational** for Bob (t_B)
  $0 \cdot 0 + 1 \cdot 1 \geq 3 \cdot 0 + 0 \cdot 1$
- Ann thinks Bob may be irrational
  $P_A(Irrat[B]) = 0.3$, $P_A(Rat[B]) = 0.7$

	H	M
t_B	0	0.5
u_B	0.2	0.3

t_A

	H	M
t_A	0	1

t_B

	H	M
t_A	0.4	0.6

u_B

Rationality

Let $G = \langle N, \{S_i\}_{i \in N}, \{u_i\}_{i \in N} \rangle$ be a strategic game and $\mathcal{T} = \langle \{T_i\}_{i \in N}, \{\lambda_i\}_{i \in N}, S \rangle$ a type space for G.

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For each $t_i \in T_i$, we can define a probability measure $p_{t_i} \in \Delta(S_{-i})$:

$$p_{t_i}(s_{-i}) = \sum_{t_{-i} \in T_{-i}} \lambda_i(t_i)(s_{-i}, t_{-i})$$

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$$p_{t_i}(s_{-i}) = \sum_{t_{-i} \in T_{-i}} \lambda_i(t_i)(s_{-i}, t_{-i})$$

The set of states (pairs of strategy profiles and type profiles) where player i chooses **rationally** is:

$$\mathsf{Rat_i} \;:=\; \{(s_i, t_i) \mid s_i \text{ is a best response to } p_{t_i}\}$$

The event that all players are *rational* is
$\mathsf{Rat} = \{(s, t) \mid \text{ for all } i, \; (s_i, t_i) \in \mathsf{Rat_i}\}$.

Common "knowledge" of rationality

In much of this literature, "full belief" or sometimes "knowledge" is identified with probability 1.

(This is not a philosophical commitment, but rather a term of art!)

Common knowledge of rationality

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Suppose that for each i, R_i^n has been defined.

Define R^n_{-i} as follows:

$$R^n_{-i} = \{(s,t) \mid s \in S_{-i},\ t \in T_{-j}, \text{ and for each } j \neq i,\ (s_j, t_j) \in R_j^n\}.$$

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For each $n > 1$,

$$R_i^{n+1} = \{(s,t) \mid (s,t) \in R_i^n \text{ and } \lambda_i(t) \text{ assigns probability 1 to } R^n_{-i}\}$$

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Common knowledge of rationality is:

$$\bigcap_{n \geq 1} R_1^n \ \times \ \bigcap_{n \geq 1} R_2^n \times \cdots \times \bigcap_{n \geq 1} R_N^n$$

Models of graded beliefs

		B	
		L	R
A	U	2,2	0,0
	D	0,0	1,1

- Consider the state (d, r, a_3, b_3). Both a_3 and b_3 correctly believe that (i.e., assign probability 1 to) the outcome is (d, r)

$\lambda_A(a_1)$	L	R
b_1	0.5	0.5
b_2	0	0
b_3	0	0

$\lambda_A(a_2)$	L	R
b_1	0.5	0
b_2	0	0
b_3	0	0.5

$\lambda_A(a_3)$	L	R
b_1	0	0
b_2	0	0.5
b_3	0	0.5

$\lambda_B(b_1)$	U	D
a_1	0.5	0
a_2	0	0.5
a_3	0	0

$\lambda_B(b_2)$	U	D
a_1	0.5	0
a_2	0	0
a_3	0	0.5

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a_1	0	0
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a_3	0	0.5

Models of graded beliefs

		B	
		L	R
A	U	2,2	0,0
	D	0,0	1,1

- This fact is not common knowledge: a_3 assigns a 0.5 probability to Bob being of type b_2, and type b_2 assigns a 0.5 probability to Ann playing l. *Ann does not know that Bob knows that she is playing r*

$\lambda_A(a_1)$	L	R
b_1	0.5	0.5
b_2	0	0
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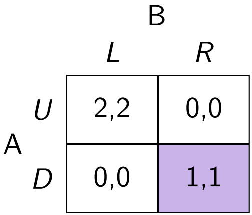
Models of graded beliefs



		B	
		L	R
A	U	2,2	0,0
	D	0,0	1,1

- Furthermore, while it is true that both Ann and Bob are rational, it is not common knowledge that they are rational.

$\lambda_A(a_1)$	L	R
b_1	0.5	0.5
b_2	0	0
b_3	0	0

$\lambda_A(a_2)$	L	R
b_1	0.5	0
b_2	0	0
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b_1	0	0
b_2	0	0.5
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$\lambda_B(b_1)$	U	D
a_1	0.5	0
a_2	0	0.5
a_3	0	0

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a_2	0	0
a_3	0	0.5

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General Comments

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- "Impossibility" is identified with probability 0, but it is an important distinction (especially for infinite games)
- We can model "soft" information using conditional probability systems, lexicographic probabilities, nonstandard probabilities (more on this later).

Models of all-out attitudes.

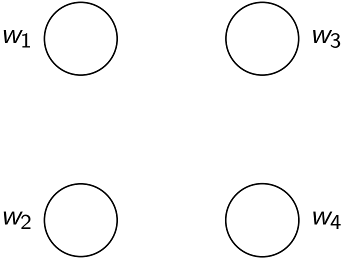
Hard Information

Example

		Bob	
		r	l
Ann	u	1, -1	-1, 1
	d	-1,1	1, -1

Example

		Bob	
		r	l
Ann	u	1, -1	-1, 1
	d	-1,1	1, -1



Example

		Bob	
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Ann	u	1, -1	-1, 1
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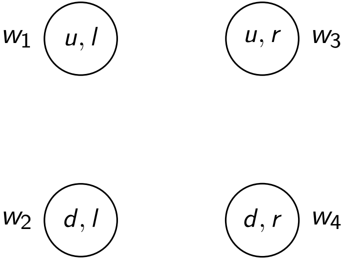

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Ann	u	1, -1	-1, 1
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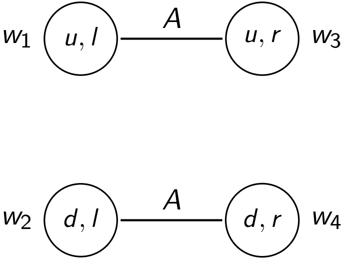

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		Bob	
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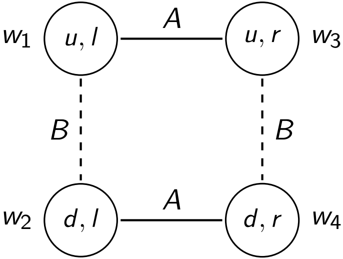

Epistemic Model

Suppose that G is a strategic game, S is the set of strategy profiles of G, and Ag is the set of players. An **epistemic model based on S and Ag** is a triple $\langle W, \{\Pi_i\}_{i\in \mathrm{Ag}}, \sigma\rangle$, where W is a nonempty set, for each $i \in \mathrm{Ag}$, Π_i is a partition[2] over W and $\sigma : W \to S$ is a strategy function.

---

[2] A partition of W is a pairwise disjoint collection of subsets of W whose union is all of W. Elements of a partition Π on W are called **cells**, and for $w \in W$, let $\Pi(w)$ denote the cell of Π containing w.

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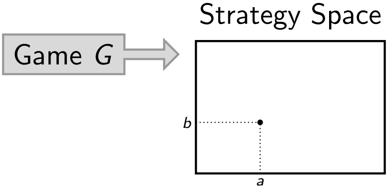
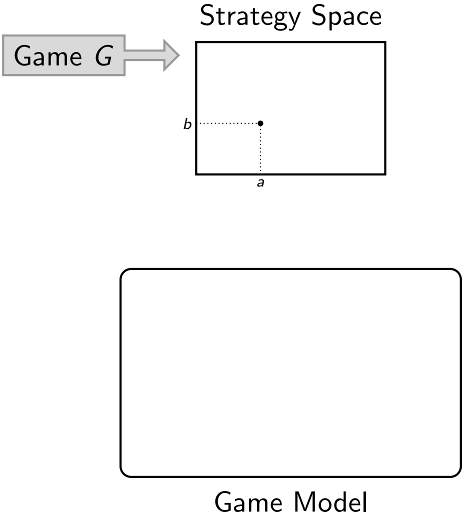
Game models

Game G

Game models



Game models



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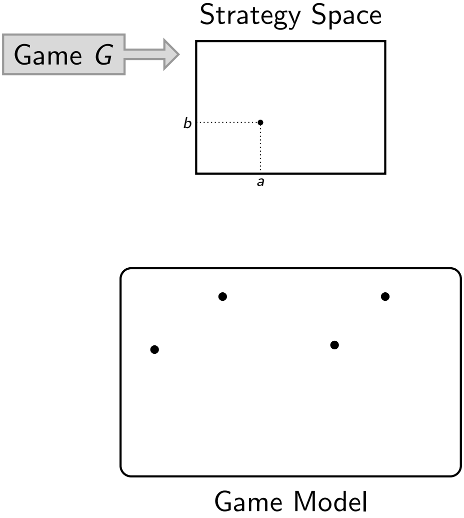

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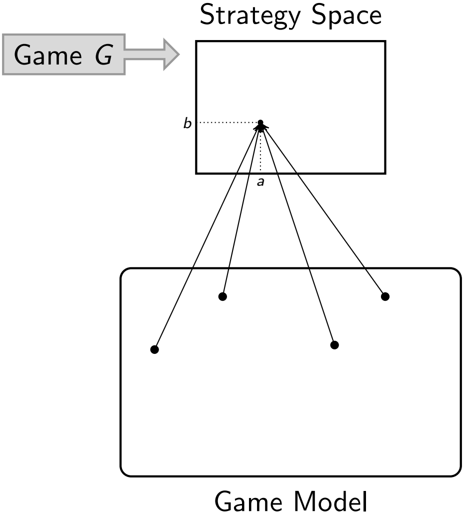
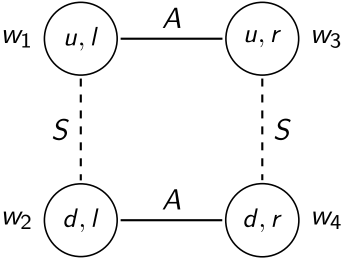


Kripke Model for S5

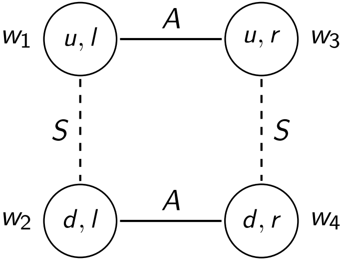
Prop is a given set of atomic propositions and Ag is a set of agents. An **epistemic model based on *Prop* and** Ag is a triple $\langle W, \{\Pi_i\}_{i\in\mathrm{Ag}}, V\rangle$, where W is a nonempty set, for each $i \in \mathrm{Ag}$, Π_i is a partition over W and $V : W \to \mathcal{P}(\mathit{Prop})$ is a valuation function.

Example

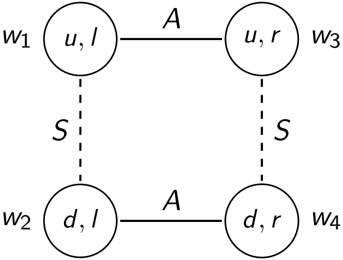
		A	
		r	l
S	u	1, -1	-1, 1
	d	-1,1	1, -1



Example

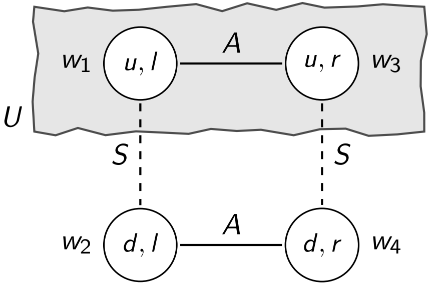


Example



- $\mathcal{M}, w \models K_i\varphi$ iff for all $w' \in \pi_i(w)$, $\mathcal{M} w' \models \varphi$.

Example



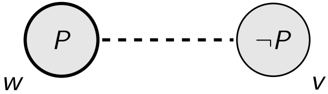
- $\mathcal{M}, w \models K_i\varphi$ iff for all $w' \in \pi_i(w)$, $\mathcal{M} w' \models \varphi$.
- One assumption: Ex-interim condition.
    - If $w' \in \pi_i(w)$ then $\sigma(w)_i = \sigma(w')_i$.

Hard Information, Axiomatically

1. Closed under known implication (K): $K_i(\varphi \rightarrow \psi) \rightarrow (K_i\varphi \rightarrow K_i\psi)$
2. Logical truth are known (NEC): If $\models \varphi$ then $\models K_i\varphi$
3. Truthful, (T): $K_i\varphi \rightarrow \varphi$
4. Positive introspection (4): $K_i\varphi \rightarrow K_iK_i\varphi$
5. Negative introspection (5): $\neg K_i\varphi \rightarrow K_i\neg K_i\varphi$

Soft Information

Modeling soft attitudes



Ann does not know that P

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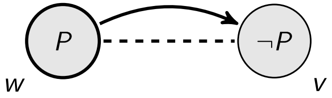

Ann does not know that P, but she believes that $\neg P$

Plausibility Models

Let *Prop* be a countable set of propositions and Ag a set of agents.
A **plausibility model** $\mathcal{M}$ is a tuple $\langle W, \{\preceq_i\}_{i\in\mathrm{Ag}}, V\rangle$ where:

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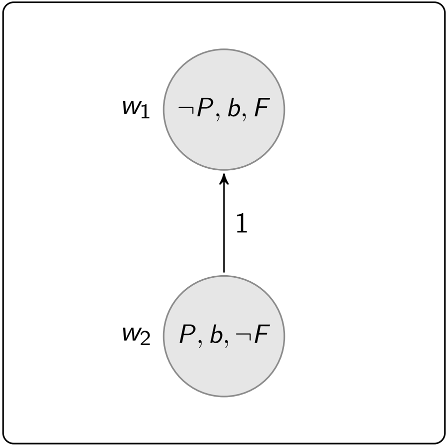
- Maximum plausibility in a given set X:
  - $max_{\preceq_i}(X) = \{w \in X : \text{for all } w' \in X, w' \preceq_i w\}$
- Hard information defined:
  - $w \sim_i w'$ iff either $w' \preceq_i w$ or $w \preceq_i w'$.
  - Let $\pi_i(w) = \{w' : w \sim_i w'\}$. Then $\{\pi_i(w) : w \in W\}$ is a partition of W.

Example - Tweety is a penguin



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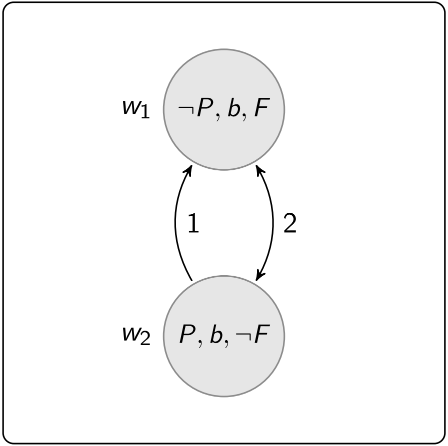

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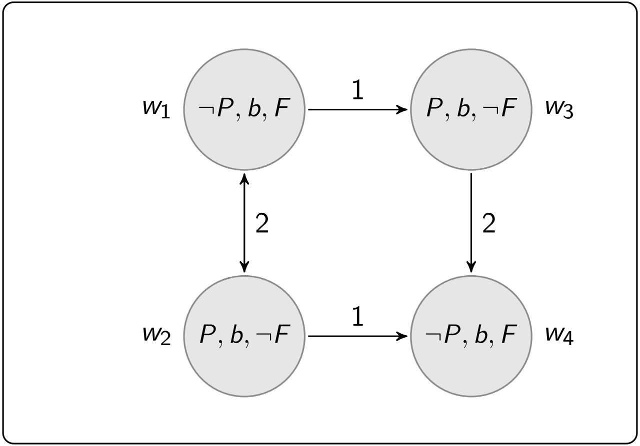

Final Remarks

- Two broad families of models of higher-order information:
  - Probabilistic/graded.
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  2. What the agents know and/or believe about 1.
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  4. ...

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- Tomorrow: we put all this machinery to work in the context of games.
- Tonight: don't miss the evening lecture.