Social Choice Theory for Logicians

Lecture 4

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Plan

- ✓ Arrow, Sen, Muller-Satterthwaite
- ✓ Characterizing Voting Methods: Majority (May, Asan & Sanver), Scoring Rules (Young), Borda Count (Farkas and Nitzan, Saari), Approval Voting (Fishburn)
- ✓ Voting to get things "right" (Distance-based measures, Condorcet and extensions)
- ✓ Strategizing (Gibbard-Satterthwaite)

1. Generalizations
   1.1 Infinite Populations
   1.2 Judgement aggregation (List & Dietrich)
2. Logics
3. Applications

Consider 3 votes, each with a confidence level $p = 2/3$.

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$$= 3 * 4/27 + 1 * 8/27$$

$$= 20/27$$

Condorcet Jury Theorem

State of the world **x** takes values 0 and 1

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M_n is the event that a majority of *n* member electorate votes correctly.

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Condorcet Jury Theorem. Suppose Independence and Competence. As the group size increases, the probability $Pr(M_n)$ that a majority votes correctly (i) increases and (ii) converges to one.

D. Austen-Smith and J. Banks. *Aggregation, Rationality and the Condorcet Jury Theorem*. The American Political Science Review, 90, 1, pgs. 34 - 45, 1996.

D. Estlund. *Opinion Leaders, Independence and Condorcet's Jury Theorem*. Theory and Decision, 36, pgs. 131 - 162, 1994.

F. Dietrich. *The premises of Condorcet's Jury Theorem are not simultaneously justified*. Episteme, 2008.

Judgement Aggregation

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Preference aggregations vs. *judgement aggregation*

- Judgements of preference, value judgements, beliefs

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Preference aggregations vs. *judgement aggregation*

- Judgements of preference, value judgements, beliefs
- What should be done? What is the best alternative?
- The Pareto conditions (see forthcoming work by W. Rabinowicz, S. Hartmann and S. Rafiee Rad)

Doctrinal Paradox

Suppose that three experts *independently* formed opinions about three propositions. For example,

1. p: "Carbon dioxide emissions are above the threshold x"
2. $p \rightarrow q$: "If carbon dioxide emissions are above the threshold x, then there will be global warming"
3. q: "There will be global warming"

Doctrinal Paradox

	p	$p \to q$	q
Expert 1			
Expert 2			
Expert 3			

Doctrinal Paradox

	p	$p \to q$	q
Expert 1	True	True	
Expert 2			
Expert 3			

Doctrinal Paradox

	p	$p \rightarrow q$	q
Expert 1	True	True	True
Expert 2			
Expert 3			

Doctrinal Paradox

	p	$p \to q$	q
Expert 1	True	True	True
Expert 2	True		False
Expert 3			

Doctrinal Paradox

	p	$p \rightarrow q$	q
Expert 1	True	True	True
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Doctrinal Paradox

	p	$p \to q$	q
Expert 1	True	True	True
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	p	$p \to q$	q
Expert 1	True	True	True
Expert 2	True	False	False
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Group			

Doctrinal Paradox

	p	$p \to q$	q
Expert 1	True	True	True
Expert 2	True	False	False
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Group	True		

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Expert 1	True	True	True
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Group	True	True	False

The Logic of Group Decisions, II

(Kornhauser and Sager 1993)

p: a valid contract was in place

q: there was a breach of contract

r: the court is required to find the defendant liable.

	p	q	$(p \wedge q) \leftrightarrow r$	r
1	yes	yes	yes	yes
2	yes	no	yes	no
3	no	yes	yes	no

The Logic of Group Decisions, II

(Kornhauser and Sager 1993)

Should we accept r?

	p	q	$(p \wedge q) \leftrightarrow r$	r
1	yes	yes	yes	yes
2	yes	no	yes	no
3	no	yes	yes	no

The Logic of Group Decisions, II

(Kornhauser and Sager 1993)

Should we accept r? No, a simple majority votes no.

	p	q	$(p \wedge q) \leftrightarrow r$	r
1	yes	yes	yes	yes
2	yes	no	yes	no
3	no	yes	yes	no

The Logic of Group Decisions, II

(Kornhauser and Sager 1993)

Should we accept r? Yes, a majority votes yes for p and q and $(p \wedge q) \leftrightarrow r$ is a legal doctrine.

	p	q	$(p \wedge q) \leftrightarrow r$	r
1	yes	yes	yes	yes
2	yes	no	yes	no
3	no	yes	yes	no

Many Variants!

See
`http://personal.lse.ac.uk/LIST/doctrinalparadox.htm`
for many generalizations!

Kornhauser and Sager. *Unpacking the court*. Yale Law Journal, 1986.

C. List and P. Pettit. *Aggregating Sets of Judgments: An Impossibility Result*. Economics and Philosophy 18: 89-110, 2002.

The Judgement Aggregation Model: The Propositions

Propositions: Let $\mathcal{L}$ be a logical language (called **propositions** in the literature) with the usual boolean connectives.

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Consistency: The standard notion of logical consistency.

Aside: We actually need

1. $\{p, \neg p\}$ are inconsistent
2. all subsets of a consistent set are consistent
3. $\emptyset$ is consistent and each $S \subseteq \mathcal{L}$ has a consistent maximal extension (not needed in all cases)

The Judgement Aggregation Model: The Agenda

Definition The **agenda** is a non-empty set $X \subseteq \mathcal{L}$, interpreted as the set of propositions on which judgments are made (note: X is a union of proposition-negation pairs $\{p, \neg p\}$).

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Example: In the discursive dilemma:
$X = \{a, \neg a, b, \neg b, a \rightarrow b, \neg(a \rightarrow b)\}$.

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Definition: Given an agenda X, each individual i's judgement set is a subset $A_i \subseteq X$.

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Rationality Assumptions:

1. A_i is **consistent**
2. A_i is **complete**, if for each $p \in X$, either $p \in A_i$ or $\neg p \in A_i$

The Judgement Aggregation Model: Aggregation Rules

Let X be an agenda, $N = \{1, \ldots, n\}$ a set of voters, a **profile** is a tuple $(A_i, \ldots, A_n)$ where each A_i is a judgement set. An **aggregation function** is a map from profiles to judgment sets. I.e., $F(A_1, \ldots, A_n)$ is a judgement set.

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Examples:

- **Propositionwise majority voting**: for each $(A_1, \ldots, A_n)$,

$$F(A_1, \ldots, A_n) = \{p \in X \mid |\{i \mid p \in A_i\}| \geq |\{i \mid p \notin A_i\}|\}$$

- **Dictator of** i: $F(A_1, \ldots, A_n) = A_i$
- **Reverse Dictator of** i: $F(A_1, \ldots, A_n) = \{\neg p \mid p \in A_i\}$

The Judgement Aggregation Model: Input Condition

Universal Domain: The domain of F is the set of all possible profiles of consistent and complete judgement sets.

The Judgement Aggregation Model: Output Condition

Collective Rationality: F generates consistent and complete collective judgment sets.

The Judgement Aggregation Model: Responsiveness Conditions

Systematicity: For any $p, q \in X$ and all $(A_1, \ldots, A_n)$ and $(A_1^*, \ldots, A_n^*)$ in the domain of F,

$$\text{if [for all } i \in N,\ p \in A_i \text{ iff } q \in A_i^*]$$
$$\text{then } [p \in F(A_1, \ldots, A_n) \text{ iff } q \in F(A_1^*, \ldots A_n^*)\ ].$$

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Independence: For any $p \in X$ and all $(A_1, \ldots, A_n)$ and $(A_1^*, \ldots, A_n^*)$ in the domain of F,

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The Judgement Aggregation Model: Responsiveness Conditions

Anonymity: For all profiles $(A_1, \ldots, A_n)$, $F(A_1, \ldots, A_n) = F(A_{\pi(1)}, \ldots, A_{\pi(n)})$ where π is a permutation of the voters.

Unanimity: For all profiles $(A_1, \ldots, A_n)$ if $p \in A_i$ for each i then $p \in F(A_1, \ldots, A_n)$

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Monotonicity: For any $p \in X$ and all $(A_1, \ldots A_i, \ldots, A_n)$ and $(A_1, \ldots, A_i^*, \ldots, A_n)$ in the domain of F,

$$\text{if } [p \notin A_i,\ p \in A_i^* \text{ and } p \in F(A_1, \ldots, A_i, \ldots A_n)]$$
$$\text{then } [p \in F(A_1, \ldots, A_i^*, \ldots A_n)].$$

The Judgement Aggregation Model: Responsiveness Conditions

Non-dictatorship: There exists no $i \in N$ such that, for any profile $(A_1, \ldots, A_n)$, $F(A_1, \ldots, A_n) = A_i$

Baseline Result

Theorem (List and Pettit, 2001) If $X \subseteq \{a, b, a \wedge b\}$, there exists no aggregation rule satisfying universal domain, collective rationality, systematicity and anonymity.

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See `personal.lse.ac.uk/LIST/doctrinalparadox.htm` for many generalizations!

Agenda Richness

Whether or not judgment aggregation gives rise to serious impossibility results depends on how the propositions in the agenda are *interconnected*.

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Definition A set $Y \subseteq \mathcal{L}$ is **minimally inconsistent** if it is inconsistent and every proper subset $X \subsetneq Y$ is consistent.

Agenda Richness

Definition An agenda X is **minimally connected** if

1. (non-simple) it has a minimal inconsistent subset $Y \subseteq X$ with $|Y| \geq 3$
2. (*even-number-negatable*) it has a minimal inconsistent subset $Y \subseteq X$ such that

$$Y - Z \cup \{\neg z \mid z \in Z\} \text{ is consistent}$$

for some subset $Z \subseteq Y$ of even size.

Impossibility Theorems

Theorem (Dietrich and List, 2007) If (and only if) an agenda is non-simple and even-number negatable, every aggregation rule satisfying universal domain, collective rationality, systematicity and unanimity is a dictatorship (or inverse dictatorship).

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Theorem (Nehring and Puppe, 2002) If (and only if) an agenda is non-simple, every aggregation rule satisfying universal domain, collective rationality, systematicity unanimity, and monotonicity is a dictatorship.

Characterization Result

$p \in X$ conditionally entails $q \in X$, written $p \vdash^* q$ provided there is a subset $Y \subseteq X$ consistent with each of p and $\neg q$ such that $\{p\} \cup Y \vdash q$.

Totally Blocked: X is totally blocked if for any $p, q \in X$ there exists $p_1, \ldots, p_k \in X$ such that

$$p = p_1 \vdash^* p_2 \vdash^* \cdots \vdash^* p_k = q$$

Characterization Result

Theorem (Dietrich and List, 2007, Dokow Holzman 2010) If (and only if) an agenda is totally blocked and even-number negatable, every aggregation rule satisfying universal domain, collective rationality, independence and unanimity is a dictatorship.

Theorem (Nehring and Puppe, 2002, 2010) If (and only if) an agenda is totally blocked, every aggregation rule satisfying universal domain, collective rationality, independence unanimity, and monotonicity is a dictatorship.

Many Variants!

Christian List. *The Theory of Judgement Aggregation: A Survey*. Synthese, forthcoming.

How should we aggregate judgements *without* independence?

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- Premiss-based aggregation
- Distance-based

What is a premiss?

An employee-owned bakery must decide whether to buy a pizza oven (P) or a fridge to freeze their outstanding Tiramisu (F). The pizza oven and the fridge cannot be in the same room. So they also need to decide whether to rent an extra room in the back (R). They all agree that they will rent the room if they decide to buy both the pizza oven and the fridge: $((P \wedge F) \rightarrow R)$, but they are contemplating renting the room regardless of the outcome of the vote on the appliances.

F. Cariani. *Judgement Aggregation*. Philosophy Compass, 6, 1, pgs. 22 - 32, 2011.

Distance-Based Aggregation

G. Pigozzi. *Belief merging and the discursive dilemma: an argument-based account of paradoxes in judgement aggregation*. Synthese 152, pgs. 285 - 298, 2006.

M. Miller and D. Osherson. *Methods for distance-based judgement aggregation*. Social Choice and Welfare, 32, pgs. 575 - 601, 2009.

C. Duddy and A. Piggins. *A measure of distance between judgement sets*. Manuscript, 2011.

Given $(A_1, \ldots, A_n)$, select the set consistent and complete A that minimizes the total distance from the individual judgement sets: find A such that $\sum_{i \in N} d(A, A_i)$ is minimized.

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Hamming Metric: $d(A, A') =$ the number of propositions for which A and A' disagree

$d_H(\{p, q, p \wedge q\}, \{p, \neg q, \neg(p \wedge q)\}) = 2$

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$d_H(\{p, q, p \wedge q\}, \{p, \neg q, \neg(p \wedge q)\}) = 2$

Duddy and Piggins: shouldn't
$d(\{p, q, p \wedge q\}, \{p, \neg q, \neg(p \wedge q)\} = 1$?

Duddy and Piggins Measure

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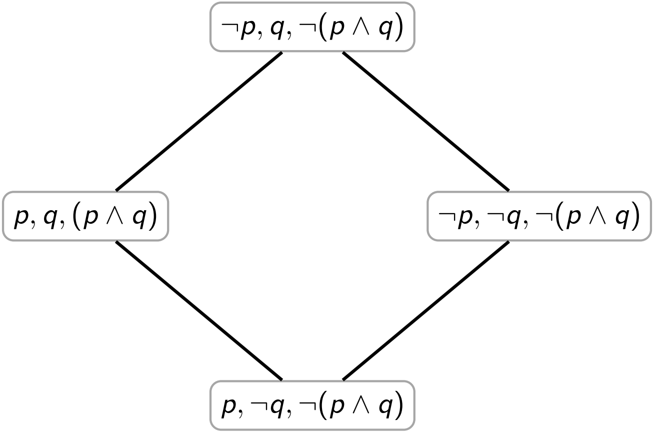
Draw a graph where the nodes are possible judgement sets and there is an edge between A and B provided there is no judgement set between them.

The distance between A and B is the length of the shortest path from A to B.


$\neg p, q, \neg(p \wedge q)$
$p, q, (p \wedge q)$
$\neg p, \neg q, \neg(p \wedge q)$
$p, \neg q, \neg(p \wedge q)$

Axioms

Axiom 1 $d(A, B) = 0$ iff $A = B$
Axiom 2 $d(A, B) = d(B, A)$
Axiom 3 $d(A, B) \leq d(A, C) + d(C, B)$

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$d(A, B) = d(A, C) + d(C, B)$

Axiom 5 If there is no judgement set between A and B with $A \neq B$ then $d(A, B) = 1$

Theorem (Duddy & Piggins) The previously defined metric is the unique metric satisfying Axioms 1 - 5.

	p	q	$p \wedge q$

	p	q	$p \wedge q$
1	T	T	T

	p	q	$p \wedge q$
1	T	T	T
2	T	F	F

	p	q	$p \wedge q$
1	T	T	T
2	T	F	F
3	F	T	F

	p	q	$p \wedge q$
1	T	T	T
2	T	F	F
3	F	T	F
Majority	T	T	F

	p	q	$p \wedge q$
1	T	T	T
2	T	F	F
3	F	T	F
Majority	T	T	F
DP-metric	T	T	T

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1	T	T	T
2	T	F	F
3	F	T	F
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DP-metric	T	T	T
Hamming	F	T	F

	p	q	$p \wedge q$
1	T	T	T
2	T	F	F
3	F	T	F
Majority	T	T	F
DP-metric	T	T	T
Hamming	F	T	F
Premise	T	T	T

M. Miller and D. Osherson. *Methods for distance-based judgement aggregation*. Social Choice and Welfare, 32, pgs. 575 - 601, 2009.

Differing on $\{a, b \wedge c\}$ may be considered more consequential than differing on $\{a, a \wedge b\}$.

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Let $\mathcal{F}$ be the set of *all* judgement sets and $\mathcal{F}^\circ$ the set of all consistent judgement sets.

$d : \mathcal{F} \times \mathcal{F} \to \mathbb{R}$

Axiom 1 $d(A, B) = 0$ iff $A = B$
Axiom 2 $d(A, B) = d(B, A)$
Axiom 3 $d(A, B) \leq d(A, C) + d(C, B)$

$d(J, J') = \sum_{i \leq n} d(J_i, J'_i)$

For a profile P, $M(P) \in \mathcal{F}$ the judgement set resulting from majority rule. P is majority consistent provided $M(P) \in \mathcal{F}^\circ$

Fix a metric d and a profile $J \in \mathcal{F}^\circ$

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- $\mathit{Full}_d(J)$ is the collection of $M(J') \in \mathcal{F}^\circ$ such that J' minimizes $d(J, J')$ over all majority consistent profiles J' in $\mathcal{F}^\circ$

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- $\mathit{Output}_d(J)$ is the collection of $M(J') \in \mathcal{F}^\circ$ such that J' minimizes $d(J, J')$ over all majority consistent profiles J' in $\mathcal{F}$ (*allowing inconsistencies*)

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- $\mathit{Endpoint}_d(J)$ is the collection of $K \in \mathcal{F}^\circ$ that minimize $d(J, J')$ over all majority consistent profiles J'

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- $\mathit{Output}_d(J)$ is the collection of $M(J') \in \mathcal{F}^\circ$ such that J' minimizes $d(J, J')$ over all majority consistent profiles J' in $\mathcal{F}$ (*allowing inconsistencies*)
- $\mathit{Endpoint}_d(J)$ is the collection of $K \in \mathcal{F}^\circ$ that minimize $d(J, J')$ over all majority consistent profiles J'
- $\mathit{Prototype}_d(J)$ is the collection of $K \in \mathcal{F}^\circ$ that minimize $\sum_{i \leq n} d(J_i, K)$ over all $K \in \mathcal{F}^\circ$

For J, K let $Ham(J, K)$ denote the Hamming distance (the number of items on which J and K disagree)

$$d(J, K) = \begin{cases} 0.9 & \text{if } J \text{ and } K \text{ disagree only on } a \wedge b \\ \sqrt{Ham(p, q)} & \text{otherwise} \end{cases}$$

	a	b	$a \wedge b$	a	b	$a \wedge b$	a	b	$a \wedge b$
1	T	T	T	T	T	T	T	T	T
2	T	T	T	T	T	T	T	T	T
3	T	F	F	T	F	F	**T**	**F**	**T**
4	T	F	F	T	F	F	T	F	F
5	F	T	F	**F**	**F**	**F**	F	T	F
M	T	T	F	T	F	F	T	T	T

	a	b	$a \wedge b$	a	b	$a \wedge b$	a	b	$a \wedge b$
1	T	T	T	T	T	T	T	T	T
2	T	T	T	T	T	T	T	T	T
3	T	F	F	T	F	F	**T**	**F**	**T**
4	T	F	F	T	F	F	T	F	F
5	F	T	F	**F**	**F**	**F**	F	T	F
M	T	T	F	T	F	F	T	T	T

- $Full_d(J) = TFF \; (d(FTF, FFF) = 1)$

	a	b	$a \wedge b$	a	b	$a \wedge b$	a	b	$a \wedge b$
1	T	T	T	T	T	T	T	T	T
2	T	T	T	T	T	T	T	T	T
3	T	F	F	T	F	F	**T**	**F**	**T**
4	T	F	F	T	F	F	T	F	F
5	F	T	F	**F**	**F**	**F**	F	T	F
M	T	T	F	T	F	F	T	T	T

- $Full_d(J) = TFF$ $(d(FTF, FFF) = 1)$
- $Output_d(J) = TTT$ $(d(TFF, TFT) = 0.9)$

	a	b	$a \wedge b$	a	b	$a \wedge b$	a	b	$a \wedge b$
1	T	T	T	T	T	T	T	T	T
2	T	T	T	T	T	T	T	T	T
3	T	F	F	T	F	F	**T**	**F**	**T**
4	T	F	F	T	F	F	T	F	F
5	F	T	F	**F**	**F**	**F**	F	T	F
M	T	T	F	T	F	F	T	T	T

- $Full_d(J) = TFF\ (d(FTF, FFF) = 1)$
- $Output_d(J) = TTT\ (d(TFF, TFT) = 0.9)$
- $Endpoint_d(J) = TTT\ (d(TTF, TTT) = 0.9)$

	a	b	$a \wedge b$	a	b	$a \wedge b$	a	b	$a \wedge b$
1	T	T	T	T	T	T	T	T	T
2	T	T	T	T	T	T	T	T	T
3	T	F	F	T	F	F	**T**	**F**	**T**
4	T	F	F	T	F	F	T	F	F
5	F	T	F	**F**	**F**	**F**	F	T	F
M	T	T	F	T	F	F	T	T	T

- $Full_d(J) = TFF$ $(d(FTF, FFF) = 1)$
- $Output_d(J) = TTT$ $(d(TFF, TFT) = 0.9)$
- $Endpoint_d(J) = TTT$ $(d(TTF, TTT) = 0.9)$
- $Prototype_d(J) = \{TTT, TFF\}$ $(\sum_i d(J_i, TTT) = 3\sqrt{2}$, $\sum_i d(J_i, TFF) = 3\sqrt{2}$, $\sum_i d(J_i, FTF) = 4\sqrt{2}$, $\sum_i d(J_i, FFF) = 2\sqrt{3} + 3)$

Tomorrow: Logic!