1. Three men are standing on a ladder, each wearing a hat. Each can see the colors of the hats of the people below him, but not his own or those higher up. It is common knowledge that only the colors red and white occur, and that there are more white hats than red ones. The actual order is white, red, white from top to bottom. Draw the epistemic model. The top person says: I know the color of my hat. Is that true? Draw the update. Who else knows his color now? If that person announces that he knows his color, what does the bottom person learn?

2. We have argued that $K_i\varphi \to K_j\varphi$ is valid on a frame $\langle W, \{R_i\}_{i\in\mathcal{A}}\rangle$ iff for each $i, j \in \mathcal{A}$, $R_j \subseteq R_i$. Find a property on frames $\langle W, \{R_i\}_{i\in\mathcal{A}}\rangle$ that guarantees that $K_i\varphi \to K_iK_j\varphi$ is valid.

3. Recall that an **epistemic-plausibility model** is a tuple

$$\mathcal{M} = \langle W, \{\sim_i\}_{i\in\mathcal{A}}, \{\preceq_i\}_{i\in\mathcal{A}}, V\rangle$$

where W is a non-empty set of states, for each $i \in \mathcal{A}$, $\sim_i$ is an equivalence relation on W, for each $i \in \mathcal{A}$, $\preceq_i$ is reflexive, transitive, and *well-founded* (every subset $X \subseteq W$ has a $\preceq_i$-minimal element), and $V : \mathsf{At} \to \wp(W)$ is a valuation function. In addition, the following two properties are satisfied:

(a) *plausibility implies possibility*: if $w \preceq_i v$ then $w \sim_i v$.

(b) *locally-connected*: if $w \sim_i v$ then either $w \preceq_i v$ or $v \preceq_i w$.

Let $\mathcal{M}$ be an epistemic-plausibility model and define $K_w = \{\varphi \mid \mathcal{M}, w \models B_i\varphi\}$. Define a revision operator $*$ as follows: $K_w * \psi = \{\varphi \mid \mathcal{M}, w \models B^{\psi}\varphi\}$. Prove that $*$ satisfies the AGM postulates ($K * 7$ and $K * 8$).

Hint: We first need some notation. Let $Ł_0$ be the sublanguage consisting of only propositional formulas (the formulas do not contain any belief/knowledge modalities). Note that the previous definitions assume that the ψ and elements of K_w are restricted to the propositional language $Ł_0$.

Given a set of sentences $X \subseteq Ł_0$ and an epistemic-plausibility model $\mathcal{M}$, we are interested in the *local consequences of* X at a state w in model $\mathcal{M}$. This is defined as follows: suppose that $[\![X]\!]_{\mathcal{M},w} = \bigcap_{\alpha\in X}[\![\alpha]\!]_{\mathcal{M}} \cap [w]$, then define

$$Cn_{\mathcal{M},w}(X) = \{\alpha \mid [\![X]\!]_{\mathcal{M},w} \subseteq [\![\alpha]\!]_{\mathcal{M},w}\}$$

To see the need for this local definition, suppose that $W = \{w_1, w_2, w_3\}$ with $w_1 \sim w_2$ (so $[w_1] = [w_2] = \{w_1, w_2\}$, but w_3 is not in this equivalence cell). Suppose that $V(p) = \{w_1, w_3\}$ and $V(q) = \{w_1\}$. Now if w_1 and w_2 are equally plausible, we have $\mathcal{M}, w_1 \models B^p q$, which implies $q \in K_{w_1} * p$. According to AGM, this means that q is a "consequence" of p. However, some care must be taken concerning what "consequence" means in this setting. Obviously, $p \to q$ is not a tautology (as p and q are different atomic propositions), furthermore we do not even have $[\![p \to q]\!]_{\mathcal{M}} = W$ (i.e., $p \to q$ is valid on the model). We only have a much weaker fact: the agent *knows* that $p \to q$ is true (i.e., it is true throughout the agents information cell at w_1). Note that this situation can also be modeled in the "standard" AGM setting: the agent is assumed to have an underlying theory which she takes as knowledge (in the previous example, the agent assumes that $p \leftrightarrow q$ is a theorem).

Note that we make use of two similar notations: $[\![\varphi]\!]_{\mathcal{M}} = \{w \mid \mathcal{M}, w \models \varphi\}$ and $[\![\varphi]\!]_{\mathcal{M},w} = [\![\varphi]\!]_{\mathcal{M}} \cap [w]$.

Fact 1. The following basic facts will be used in the proof below.

(a) $[\![\varphi \wedge \psi]\!]_{\mathcal{M}} = [\![\varphi]\!]_{\mathcal{M}} \cap [\![\psi]\!]_{\mathcal{M}}$, and so $[\![\varphi \wedge \psi]\!]_{\mathcal{M},w} = [\![\varphi]\!]_{\mathcal{M},w} \cap [\![\psi]\!]_{\mathcal{M},w}$

Proof. Immediate from the definitions: $[\![\varphi \wedge \psi]\!]_{\mathcal{M}} = \{w \mid \mathcal{M}, w \models \varphi \wedge \psi\} = \{w \mid \mathcal{M}, w \models \varphi \text{ and } \mathcal{M}, w \models \psi\} = \{w \mid \mathcal{M}, w \models \varphi\} \cap \{w \mid \mathcal{M}, w \models \psi\} = [\![\varphi]\!]_{\mathcal{M}} \cap [\![\psi]\!]_{\mathcal{M}}$. QED

(b) Let X be a set of formulas and φ a formula, then $[\![X \cup \{\varphi\}]\!]_{\mathcal{M},w} = [\![X]\!]_{\mathcal{M},w} \cap [\![\varphi]\!]_{\mathcal{M},w}$

Proof. Immediate from the definitions:

$$[\![X \cup \{\varphi\}]\!]_{\mathcal{M},w} = \bigcap_{\beta \in X \cup \{\varphi\}} [\![\beta]\!]_{\mathcal{M},w} = \bigcap_{\beta \in X} [\![\beta]\!]_{\mathcal{M},w} \ \cap \ [\![\varphi]\!]_{\mathcal{M},w} = [\![X]\!]_{\mathcal{M},w} \cap [\![\varphi]\!]_{\mathcal{M},w}.$$

QED

(c) $Min_{\preceq}([w] \cap [\![\varphi]\!]_{\mathcal{M}}) \subseteq [\![K_w * \varphi]\!]_{\mathcal{M},w}$ ($Min_{\preceq}([w]) \subseteq [\![K_w]\!]_{\mathcal{M},w}$ by letting φ be $\top$).

Proof. Let $\mathcal{M}$ be an epistemic-plausibility model and w a state in $\mathcal{M}$. Since for each $\beta \in K_w * \varphi$, $\mathcal{M}, w \models B^{\varphi}\beta$, we have for each $\beta \in K_w * \varphi$, $Min_{\preceq}([w] \cap [\![\varphi]\!]_{\mathcal{M}}) \subseteq [\![\beta]\!]_{\mathcal{M}}$. Since for all w, $Min_{\preceq}([w] \cap [\![\varphi]\!]_{\mathcal{M}}) \subseteq [w]$, we also have for all $\beta \in K_w * \varphi$, $Min_{\preceq}([w] \cap [\![\varphi]\!]_{\mathcal{M}}) \subseteq [\![\beta]\!]_{\mathcal{M}} \cap [w] = [\![\beta]\!]_{\mathcal{M},w}$. Hence,

$$Min([w] \cap [\![\varphi]\!]_{\mathcal{M}}) \subseteq \bigcap_{\beta \in K_w * \varphi} [\![\beta]\!]_{\mathcal{M},w} = [\![K_w * \varphi]\!]_{\mathcal{M},w}$$

QED

(d) If $Min_{\preceq}([w] \cap [\![\varphi]\!]_{\mathcal{M}}) \subseteq [\![\alpha]\!]_{\mathcal{M}}$, then $[\![K_w * \varphi]\!]_{\mathcal{M},w} \subseteq [\![\alpha]\!]_{\mathcal{M},w}$

Proof. Suppose that $Min_{\preceq}([w] \cap [\![\varphi]\!]_{\mathcal{M}}) \subseteq [\![\alpha]\!]_{\mathcal{M}}$. Then, $\mathcal{M}, w \models B^{\varphi}\alpha$. Hence, $\alpha \in K_w * \varphi$. This implies $[\![K_w * \varphi]\!]_{\mathcal{M},w} \subseteq [\![\alpha]\!]_{\mathcal{M},w}$ (since α is one of the conjuncts in $[\![K_w * \varphi]\!]_{\mathcal{M},w}$.

QED

(e) If $Min_{\preceq}([w] \cap [\![\varphi]\!]_{\mathcal{M}}) \cap [\![\psi]\!]_{\mathcal{M}} \subseteq [\![\alpha]\!]_{\mathcal{M}}$ then $[\![K_w * \varphi]\!]_{\mathcal{M},w} \cap [\![\psi]\!]_{\mathcal{M}} \subseteq [\![\alpha]\!]_{\mathcal{M}}$

Proof. *Supply the proof.* QED

(f) For sets $X \subseteq W$ and $Y \subseteq W$, $Min_{\preceq}(X) \cap Y \subseteq Min_{\preceq}(X \cap Y)$

Proof. If $Min_{\preceq}(X) \cap Y = \emptyset$ then we are done. Suppose that $v \in Min_{\preceq}(X) \cap Y$. Then $v \in X \cap Y$. Let $y \in X \cap Y$, then $y \in X$ and since $v \in Min_{\preceq}(X)$, $v \preceq y$. Hence, $v \in Min_{\preceq}(X \cap Y)$ and so $Min_{\preceq}(X) \cap Y \subseteq Min_{\preceq}(X \cap Y)$. QED

(g) For sets $X \subseteq W$ and $Y \subseteq W$, if $Min(X) \cap Y \neq \emptyset$, then $Min_{\preceq}(X \cap Y) = Min_{\preceq}(X) \cap Y$.

Proof. *Supply the proof* QED

4. (Extra Credit) Given any epistemic-probability structure $\langle W, \{\sim_i\}_{i \in \mathcal{A}}, \{\pi_i\}_{i \in \mathcal{A}} \rangle$, we define

> i ***believes to degree at least*** p**:** $B_i^p : \wp(W) \to \wp(W)$ is defined as $B_i^p(E) = \{w \mid \pi_i(E \mid [w]_i) \geq p\}$

Let $\langle W, \{\sim_i\}_{i \in \mathcal{A}}, \pi \rangle$ be a Bayesian model with a common prior and Π_i the partition corresponding to the equivalence relation $\sim_i$ (i.e., $\Pi_i = \{[w]_i \mid w \in W\}$ where $[w]_i$ is the equivalence class of w under $\sim_i$. Prove that for all $E \subseteq W$,

(a) $B_i^p(B_i^p(E)) = B_i^p(E)$

(b) $\pi(E \mid B_i^p(E)) \geq p$

Hint: Note that $B_i^p(E) = \bigcup_{w \in B_i^p(E)} [w]_i$ (you may use this fact without proof).