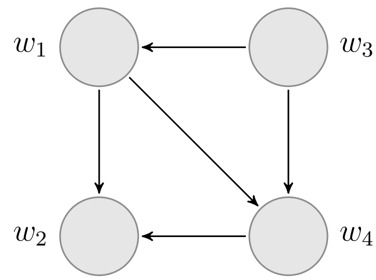
1. Consider the following relational structure (assume that there are no atomic propositions in the language):



$\mathcal{M}$

For each of the follows sets of states, find a formula that is true at precisely those sets (note that since there are no atomic propositions, the formulas will be construction using $\bot$ and $\top$): $\emptyset$, $\{w_1\}$, $\{w_2\}$, $\{w_3\}$, $\{w_4\}$, $\{w_1, w_2, w_3, w_4\}$.

2. Let $\mathcal{N} = \langle \mathbb{N}, S_1, S_2 \rangle$ and $\mathcal{B} = \langle \mathbb{B}, R_1, R_2 \rangle$ be the following frames for a language with two modalities $\Diamond_1$ and $\Diamond_2$. Here $\mathbb{N}$ is the set of natural numbers and $\mathbb{B}$ is the set of binary strings of 0s and 1s, and the relations are defined by

$$\begin{array}{rll} mS_1n & \text{iff} & n = m + 1 \\ mS_2n & \text{iff} & m > n \\ sR_1t & \text{iff} & t = s0 \text{ or } t = s1 \\ sR_2t & \text{iff} & t \text{ is a proper initial segment of } s \end{array}$$

Which of the following formulas are valid on $\mathcal{N}$ and $\mathcal{B}$, respectively?

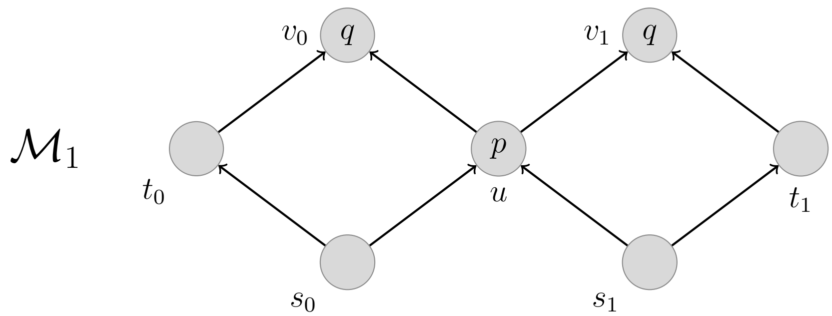
(a) $(\Diamond_1 p \wedge \Diamond_1 q) \to \Diamond_1(p \wedge q)$
(b) $(\Diamond_2 p \wedge \Diamond_2 q) \to \Diamond_2(p \wedge q)$
(c) $p \to \Diamond_1 \Box_2 p$
(d) $p \to \Diamond_2 \Box_1 p$
(e) $p \to \Box_1 \Diamond_2 p$
(f) $p \to \Box_2 \Diamond_1 p$

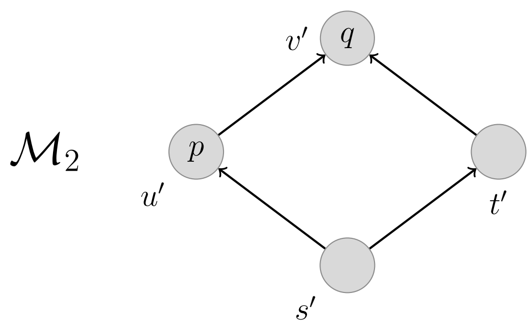
3. Suppose we wanted an operator D with the following truth definition: for any model $\mathcal{M}$ and formula φ, $\mathcal{M}, w \models D\varphi$ iff there is a $u \neq w$ such that $\mathcal{M}, u \models \varphi$. This operator is called the difference operator. Is the difference operator definable in the basic modal language? If so, give the definition and if not, give a proof.

4. Consider the binary until operator U. In a model $\mathcal{M} = \langle W, R, V\rangle$ its truth definition reads:

   $\mathcal{M}, w \models U(\varphi, \psi)$ iff there is a v such that wRv and $\mathcal{M}, v \models \varphi$, and for all x such that wRx and xRv, $\mathcal{M}, x \models \psi$.

   Prove that U is not definable in the basic modal language. Hint: consider the following two models (with extra arrows added to ensure that the relations are transitive).





5. Let $\mathcal{M} = \langle W, R, V\rangle$ be a model. Recall the definitions of the m and l, for any $X \subseteq W$,

$$m(X) = \{w \mid \text{there is a } v \text{ such that } wRv \text{ and } v \in X\}$$

   and

$$l(X) = \{w \mid \text{ for all } v, \text{ if } wRv \text{ then } v \in X\}.$$

   An ultrafilter extension is a model $ue(\mathcal{M}) = \langle \mathsf{Uf}(W), R^{ue}, V^{ue}\rangle$ where $\mathsf{Uf}(W) = \{u \mid u \text{ is an ultrafilter over } W\}$, $uR^{ue}u'$ iff for all $X \subseteq W$, if $X \in u'$ then $m(X) \in u$, and $V(p) = \underline{\{u \mid V(p) \in u\}}$. Prove that $uR^{ue}v$ iff $\{Y \mid l(Y) \in u\} \subseteq v$ (Hint: first prove that $l(X) = \overline{m(\overline{X})}$, where $\overline{X}$ is the complement of X in W).

The homework is DUE Tuesday, March 6, 2012.