1. Let $\mathcal{M} = \langle W, \leq, V\rangle$ be a preference model (as in the paper by Johan van Benthem, Patrick Girard and Olivier Roy on Preference Logic). Define $\mathcal{M}^{\#\varphi} = \langle W', \leq', V'\rangle$ as follows: $W' = W$, $\leq' = \leq - \{(w, v) \mid \mathcal{M}, w \models \varphi \text{ and } \mathcal{M}, v \models \neg\varphi\}$ and $V' = V$. Define $\mathcal{M}, w \models [\#\varphi]\psi$ iff $\mathcal{M}^{\#\varphi}, w \models \psi$.

   - What type of action does this model? That is, give an intuitive explanation of the '$\#\varphi$' operation (explain what the operation does to a model).
   - Prove that the following reduction axiom is valid:

$$[\#\varphi]\langle\leq\rangle\psi \leftrightarrow (\neg\varphi \wedge \langle\leq\rangle[\#\varphi]\psi) \vee (\varphi \wedge \langle\leq\rangle(\varphi \wedge [\#\varphi]\psi))$$

2. Let $\mathcal{M} = \langle W, \leq, V\rangle$ be a preference model. Recall that $\mathcal{M}, w \models \varphi \leq_{\exists\exists} \psi$ iff there is s, t such that $\mathcal{M}, s \models \varphi$ and $\mathcal{M}, t \models \psi$ and $s \leq t$. Prove that the following formula is valid:

$$(((\varphi \leq_{\exists\exists} \psi) \wedge \alpha) \leq_{\exists\exists} \beta) \leftrightarrow ((\varphi \leq_{\exists\exists} \psi) \wedge (\alpha \leq_{\exists\exists} \beta))$$

3. Let At be a set of atomic proposition and $\mathbb{S}$ a set of **reasons** (in the paper, this is the set of non-empty finite subsets of the natural numbers). A model is a tuple $\langle W, s, u, V\rangle$, where

   - W is a set of states
   - $s : W \times \mathcal{P}_{\neq\emptyset}(W) \to W$ is a selection function (with the condition $s(w, A) \in A$)
   - $u : W \times \mathbb{S} \to \mathfrak{R}$ is a utility function (write $u_X(\cdot)$ for $u(\cdot, X)$), and
   - $V : \mathsf{At} \to \mathcal{P}(W)$ is a valuation function

   Truth of the preference modality is:

$$\mathcal{M}, w \models \theta \succeq_X \psi \text{ iff } u_X(s(w, [\![\theta]\!]_{\mathcal{M}})) \geq u_X(s(w, [\![\psi]\!]_{\mathcal{M}}))$$

   provided $[\![\theta]\!]_{\mathcal{M}} \neq \emptyset$ *and* $[\![\psi]\!]_{\mathcal{M}} \neq \emptyset$

   - $\mathcal{M}$ is **regular** if $A \subseteq B$ and $w_1 \in A$ then If $s(w, B) = w_1$ then $s(w, A) = w_1$.
   - $\mathcal{M}$ is **proximal** if for all w and $A \neq \emptyset$, If $s(w, A) = w_1$ then there is no $w_2 \in A$ such that $V^{-1}(w)\Delta V^{-1}(w_2) \subset V^{-1}(w)\Delta V^{-1}(w_1)$, where Δ is the symmetric difference and $V^{-1}(\cdots)$ is the inverse of the valuation function.

   Prove that $(p \wedge ((p \wedge q) \succ_X r)) \to (q \succ_X r)$ is not valid on the class of all models, but it is valid on the class of regular and proximal models (see Sections 4.2, pgs. 11 & 12 and 5.1, pgs. 15 & 16 of the Osherson and Weinstein paper)

4. Consider the formulas 1-8 in Exercise 4.6.1 on pg. 101 of Fitting and Mendelsohn. Determine which of these formulas are valid on constant domain models, which are valid on variable domain models, which are valid on monotonic models and which are valid on anti-monotonic models.

5. Recall that a **FOIL Model** is a tuple $\langle W, R, D_O, D_I, I \rangle$ where I is an interpretation (mapping predicate symbols P and state w to appropriate relations on $D_O \cup D_I$, see the Fitting paper for details). Elements of D_I are functions from W to D_O. A substitution σ maps elements of V_O (object variables) to elements of D_O and elements of V_I to elements of D_I. Assume that '=' is in the language and is interpreted as the constant function mapping each state to the identity relation on D_O. We have:

   - $\mathcal{M}, w \models_\sigma \Box\varphi$ iff for each v if wRv then $\mathcal{M}, v \models_\sigma \varphi$
   - $\mathcal{M}, w \models_\sigma \forall x\varphi$ iff for each substitution σ' if σ' is an x-variant of σ (i.e., $\sigma'(y) = \sigma(y)$ for all $y \neq x$), then $\mathcal{M}, w \models_{\sigma'} \varphi$ (same definition applies for intension variables)
   - $\mathcal{M}, w \models_\sigma \langle \lambda x.\varphi \rangle(f)$ iff $\mathcal{M}, w \models_{\sigma'} \varphi$ where σ' is like σ except $\sigma'(x) = \sigma(f)(w)$.

   Prove that
   $$\forall f \forall g[\langle \lambda x, y.(x = y)\rangle(f, g) \to \langle \lambda x, y.\Box(x = y)\rangle(f, g)]$$
   is valid, but
   $$\forall f \forall g[\langle \lambda x, y.(x = y)\rangle(f, g) \to \Box\langle \lambda x, y.(x = y)\rangle(f, g)]$$
   is not valid