Examples of Modal Deduction
Notes on Lecture 5

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Propositional Rules:

$$\frac{\mathcal{A}, \neg\varphi \Rightarrow \mathcal{B}}{\mathcal{A} \Rightarrow \mathcal{B}, \varphi}\ \neg\text{:left} \qquad \frac{\mathcal{A} \Rightarrow \mathcal{B}, \neg\varphi}{\mathcal{A}, \varphi \Rightarrow \mathcal{B}}\ \neg\text{:right}$$

$$\frac{\mathcal{A}, \varphi \wedge \psi \Rightarrow \mathcal{B}}{\mathcal{A}, \varphi, \psi \Rightarrow \mathcal{B}}\ \wedge\text{:left} \qquad \frac{\mathcal{A} \Rightarrow \mathcal{B}, \varphi \wedge \psi}{\begin{array}{l}\mathcal{A} \Rightarrow \mathcal{B}, \varphi \\ \mathcal{A} \Rightarrow \mathcal{B}, \psi\end{array}}\ \wedge\text{:left}$$

Modal Rule:

$\{p_1, \ldots, p_k\}, \Diamond\varphi_1, \ldots, \Diamond\varphi_l \Rightarrow \{q_1, \ldots, q_m\}, \Diamond\psi_1, \ldots, \Diamond\psi_n$ is valid iff either

1. There is a p_i with $1 \leq i \leq k$ and q_j with $1 \leq j \leq m$ such that $p_i = q_j$,

   or

2. There is an i with $1 \leq i \leq l$ such that $\varphi_i \Rightarrow \psi_1, \ldots, \psi_n$ is valid

To show that a modal formula φ is valid, first translate it into a formula only containing $\Diamond$, $\neg$ and $\wedge$ by applying DeMorgan's rules and duality for $\Box$:

- $\varphi \to \psi$ is equivalent to $\neg(\varphi \wedge \neg\psi)$
- $\varphi \vee \psi$ is equivalent to $\neg(\neg\varphi \wedge \neg\psi)$
- $\Box\varphi$ is equivalent to $\neg\Diamond\neg\varphi$

After translating the formula to an equivalent formula φ', use the above rules to determine if the sequent '$\Rightarrow \varphi'$' is valid. We say that a sequent $\mathcal{A} \Rightarrow \mathcal{B}$ is **closed** provided there is an atomic proposition $p \in \mathcal{A} \cap \mathcal{B}$.

Example 1: $\Diamond(p \land q) \to (\Diamond p \land \Diamond q)$ is valid:

$$\dfrac{\Rightarrow\ \neg(\Diamond(p \land q) \land \neg(\Diamond p \land \Diamond q))}{\Diamond(p \land q) \land \neg(\Diamond p \land \Diamond q) \Rightarrow}\ \neg\text{:right}$$

$$\dfrac{\Diamond(p \land q) \land \neg(\Diamond p \land \Diamond q) \Rightarrow}{\Diamond(p \land q), \neg(\Diamond p \land \Diamond q) \Rightarrow}\ \land\text{:left}$$

$$\dfrac{\Diamond(p \land q), \neg(\Diamond p \land \Diamond q) \Rightarrow}{\Diamond(p \land q) \Rightarrow \Diamond p \land \Diamond q}\ \neg\text{:left}$$

$$\dfrac{\Diamond(p \land q) \Rightarrow \Diamond p \land \Diamond q}{\Diamond(p \land q) \Rightarrow \Diamond p \qquad \Diamond(p \land q) \Rightarrow \Diamond q}\ \land\text{:right}$$

$$\dfrac{\Diamond(p \land q) \Rightarrow \Diamond q}{p \land q \Rightarrow p \qquad p \land q \Rightarrow q}\ \Diamond$$

$$\dfrac{p \land q \Rightarrow q}{p, q \Rightarrow p \qquad p, q \Rightarrow q}\ \land\text{:left}$$

Example 2: $(\Diamond p \land \Diamond q) \to \Diamond(p \land q)$ is not valid:

$$\dfrac{\Rightarrow\ \neg((\Diamond p \land \Diamond q) \land \neg\Diamond(p \land q))}{\Diamond p \land \Diamond q) \land \neg\Diamond(p \land q) \Rightarrow}\ \neg\text{:right}$$

$$\dfrac{\Diamond p \land \Diamond q) \land \neg\Diamond(p \land q) \Rightarrow}{\Diamond p \land \Diamond q, \neg\Diamond(p \land q) \Rightarrow}\ \land\text{:left}$$

$$\dfrac{\Diamond p \land \Diamond q, \neg\Diamond(p \land q) \Rightarrow}{\Diamond p \land \Diamond q \Rightarrow \Diamond(p \land q)}\ \neg\text{:left}$$

$$\dfrac{\Diamond p \land \Diamond q \Rightarrow \Diamond(p \land q)}{\Diamond p, \Diamond q \Rightarrow \Diamond(p \land q)}\ \land\text{:left}$$

There are two possible ways to proceed with this proof:

$$\dfrac{p \Rightarrow p \land q}{p \Rightarrow p \qquad p \Rightarrow q}\ \land\text{:right}$$

which does not close since $p \Rightarrow q$ is not valid.

$$\dfrac{q \Rightarrow p \land q}{q \Rightarrow p \qquad q \Rightarrow q}\ \land\text{:right}$$

which does not close since $q \Rightarrow p$ is not valid.

Example 3: $\Box(p \wedge q) \to (\Box p \wedge \Box q)$ is valid:

$$\Rightarrow \neg(\neg\Diamond\neg(p \wedge q) \wedge \neg(\neg\Diamond\neg p \wedge \neg\Diamond\neg q))$$
$\neg$:right
$$\neg\Diamond\neg(p \wedge q) \wedge \neg(\neg\Diamond\neg p \wedge \neg\Diamond\neg q) \Rightarrow$$
$\wedge$:left
$$\neg\Diamond\neg(p \wedge q), \neg(\neg\Diamond\neg p \wedge \neg\Diamond\neg q) \Rightarrow$$
$\neg$:left
$$\neg(\neg\Diamond\neg p \wedge \neg\Diamond\neg q) \Rightarrow \Diamond\neg(p \wedge q)$$
$\neg$:left
$$\Rightarrow \Diamond\neg(p \wedge q), \neg\Diamond\neg p \wedge \neg\Diamond\neg q$$
$\wedge$:right
$$\Rightarrow \Diamond\neg(p \wedge q), \neg\Diamond\neg p$$
$$\Rightarrow \Diamond\neg(p \wedge q), \neg\Diamond\neg q$$
$\neg$:right
$$\Diamond\neg p \Rightarrow \Diamond\neg(p \wedge q)$$
$$\Diamond\neg q \Rightarrow \Diamond\neg(p \wedge q)$$
$\Diamond$
$$\neg p \Rightarrow \neg(p \wedge q)$$
$$\neg q \Rightarrow \neg(p \wedge q)$$
$$\Diamond\neg q \Rightarrow \Diamond\neg(p \wedge q)$$
$\neg$:right
$$p \wedge q, \neg p \Rightarrow$$
$$p \wedge q, \neg q \Rightarrow$$
$\neg$:left
$$p \wedge q \Rightarrow p$$
$$p \wedge q \Rightarrow q$$
$\wedge$:left
$$p, q \Rightarrow p$$
$$p, q \Rightarrow q$$