The Canonical Model
Notes for Lecture 6

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Notation:

- Let **K** denote the minimal modal logic and $\vdash \varphi$ mean φ is derivable in **K**. If Γ is a set of formulas, we write $\Gamma \vdash \varphi$ if $\vdash (\psi_1 \wedge \cdots \wedge \psi_k) \rightarrow \varphi$ for some finite set $\psi_1, \ldots, \psi_k \in \Gamma$.
- Let Γ be a set of formulas. If $\mathcal{F}$ is a frame, then we write $\mathcal{F} \models \Gamma$ for $\mathcal{F} \models \varphi$ for each $\varphi \in \Gamma$. We write $\Gamma \models \varphi$ provided for all frames $\mathcal{F}$, if $\mathcal{F} \models \Gamma$ then $\mathcal{F} \models \varphi$.
- A set of formulas Γ is **consistent** provided $\Gamma \not\vdash \bot$.
- Γ is a **maximally consistent set** if Γ is consistent and for each $\varphi \in \mathcal{L}$ either $\varphi \in \Gamma$ of $\neg\varphi \in \Gamma$. Alternatively, Γ is consistent and every Γ' such that $\Gamma \subseteq \Gamma'$ is inconsistent.
- A logic is strongly complete if $\Gamma \models \varphi$ implies $\Gamma \vdash \varphi$. It is weakly complete if $\models \varphi$ implies $\vdash \varphi$. Strong completeness implies weak completeness, but weak completeness does not imply strong completeness.

Important facts about maximally consistent sets: Suppose that Γ is a maximally consistent set,

1. If $\vdash \varphi$ then $\varphi \in \Gamma$
2. If $\varphi \rightarrow \psi \in \Gamma$ and $\varphi \in \Gamma$ then $\psi \in \Gamma$
3. $\neg\varphi \in \Gamma$ iff $\varphi \notin \Gamma$
4. $\varphi \wedge \psi \in \Gamma$ iff $\varphi \in \Gamma$ and $\psi \in \Gamma$
5. $\varphi \vee \psi \in \Gamma$ iff $\varphi \in \Gamma$ or $\psi \in \Gamma$

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Lemma 1 (Lindenbaum's Lemma) *For each consistent set* Γ*, there is a maximally consistent set* Γ' *such that* $\Gamma \subseteq \Gamma'$*. In other words, every consistent set* Γ *can be extended to a maximally consistent set.*

Definition 2 (Canonical Model) The canonical model for **K** is the model $\mathcal{M}^c = \langle W^c, R^c, V^c \rangle$ where

- $W^c = \{\Gamma \mid \Gamma \text{ is a maximally consistent set}\}$
- $\Gamma R^c \Delta$ iff $\Gamma^\Box = \{\varphi \mid \Box\varphi \in \Gamma\} \subseteq \Delta$
- $V^c(p) = \{\Gamma \mid p \in \Gamma\}$ ◁

Lemma 3 (Truth Lemma) *For every* $\varphi \in \mathcal{L}$*,* $\mathcal{M}^c, \Gamma \models \varphi$ *iff* $\varphi \in \Gamma$

Theorem 4 *Every maximally consistent set* Γ *has a model (i.e., there is a models* $\mathcal{M}$ *and state* w *such that for all* $\varphi \in \Gamma$*,* $\mathcal{M}, w \models \varphi$*.*

Proof. Suppose that Γ is a consistent set. By Lindenbaum's Lemma, there is a maximally consistent set Γ' such that $\Gamma \subseteq \Gamma'$. Then, by the Truth Lemma, for each $\varphi \in \Gamma'$, we have $\mathcal{M}^c, \Gamma' \models \varphi$. Then, in particular, every formula in Γ is true at Γ' in the canonical model. QED

Theorem 5 *If* $\Gamma \models \varphi$ *then* $\Gamma \vdash \varphi$

Proof. Suppose that $\Gamma \nvdash \varphi$. Then, $\Gamma \cup \{\neg\varphi\}$ is consistent. By the above theorem, there is a model of $\Gamma \cup \{\neg\varphi\}$. Hence, $\Gamma \not\models \varphi$. QED

Suppose that **L** is a logic extending **K**. We can build a canonical model for **L** as above. The question is: Is the canonical model in the appropriate class of models?

Lemma 6 *If* $\Box\varphi \to \varphi \in$ **L***, then the canonical model for* **L** *is reflexive.*

Proof. Suppose that $\Box\varphi \to \varphi$ is derivable in **L**. We must show that for any MCS Γ, $\Gamma R^c \Gamma$. That is, $\Gamma^\Box = \{\varphi \mid \Box\varphi \in \Gamma\} \subseteq \Gamma$. Suppose that $\Box\psi \in \Gamma$. We must show that $\psi \in \Gamma$. This follows since $\Box\psi \to \psi \in \Gamma$ and Γ is closed under modus ponens. QED

Lemma 7 *If* $\Box\varphi \to \Box\Box\varphi \in$ **L***, then the canonical model for* **L** *is transitive.*

Proof. Suppose that $\Box\varphi \to \Box\Box\varphi$ is derivable in **L**. We must show that for MCS $\Gamma, \Gamma', \Gamma''$, if $\Gamma R^c \Gamma'$ and $\Gamma' R^c \Gamma''$, then $\Gamma R^c \Gamma''$. Suppose that $\Gamma R^c \Gamma'$ and $\Gamma' R^c \Gamma''$. Then, $\{\varphi \mid \Box\varphi \in \Gamma\} \subseteq \Gamma'$ and $\{\varphi \mid \Box\varphi \in \Gamma'\} \subseteq \Gamma''$. We must show $\{\varphi \mid \Box\varphi \in \Gamma\} \subseteq \Gamma''$. Suppose that $\Box\psi \in \Gamma$. Then, since $\Box\psi \to \Box\Box\psi \in \Gamma$, we have $\Box\Box\psi \in \Gamma$. This means, $\Box\psi \in \Gamma'$ and $\psi \in \Gamma''$, as desired. QED

Theorem 8 **S4** *is sound and strongly complete with respect to the class of Kripke structures that are reflexive and transitive.*

Lemma 9 *If* $\neg\Box\varphi \to \Box\neg\Box\varphi \in \mathbf{L}$*, then the canonical model for* **L** *is Euclidean.*

Proof. Suppose that $\neg\Box\varphi \to \Box\neg\Box\varphi$ is derivable in **L**. We must show that for MCS $\Gamma, \Gamma', \Gamma''$, if $\Gamma R^c \Gamma'$ and $\Gamma R^c \Gamma''$, then $\Gamma' R^c \Gamma''$. Suppose that $\Gamma R^c \Gamma'$ and $\Gamma R^c \Gamma''$. Then, $\{\varphi \mid \Box\varphi \in \Gamma\} \subseteq \Gamma'$ and $\{\varphi \mid \Box\varphi \in \Gamma\} \subseteq \Gamma''$. We must show $\{\varphi \mid \Box\varphi \in \Gamma'\} \subseteq \Gamma''$. Suppose that $\Box\psi \in \Gamma'$. If $\psi \notin \Gamma''$, then $\neg\psi \in \Gamma''$. This implies that $\Box\psi \notin \Gamma$, and hence, $\neg\Box\psi \in \Gamma$. Since $\neg\Box\psi \to \Box\neg\Box\psi \in \Gamma$, we have $\Box\neg\Box\psi \in \Gamma$. This implies that $\neg\Box\psi \in \Gamma'$, a contradiction. Hence, $\psi \in \Gamma''$, as desired. QED

Theorem 10 **S5** *is sound and strongly complete with respect to the class of Kripke structures that are equivalence relations (reflexive, transitive and symmetric).*

Completeness-via-canonicity: Let φ be a modal formula and P a property. If every normal modal logic containing φ has property P and φ is valid on any class of frames with property P, then φ is **canonical for** P.

Limitations to the above approach:

- **Undefinable Properties**: Completeness by *transforming the canonical model*: **S4** is sound and strongly complete with respect to the class of reflexive and transitive *trees*. What is the modal logic of *strict total orders*?

- **Weak Completeness**: there are normal modal logics that are not strongly complete. Eg., **KL** (**K** plus $\Box(\Box\varphi \to \varphi) \to \Box\varphi$) is not strongly complete.

- **Incompleteness** There are *consistent* normal modal logics that are not complete with respect to any class of frames. Eg., the tense logic generated by the following axioms:

  - $Fp \wedge Fq \to (F(p \wedge Fq) \vee F(p \wedge q) \vee F(Fp \wedge q))$
  - $Gp \to Fp$
  - $H(Hp \to p) \to Hp$
  - $GFp \to FGp$

  is consistent and incomplete.