Notes from Lecture 4: Ultrafilter Extensions and the Standard Translation

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1 Ultrafilter Extensions

Definition 1.1 (Ultrafilter) Let W be a non-empty set. An ultrafilter on W is a set $u \subseteq \wp(W)$ satisfying the following properties:

1. $\emptyset \notin u$
2. If $X, Y \in u$ then $X \cap Y \in u$
3. If $X \in u$ and $X \subseteq Y$ then $Y \in u$.
4. For all $X \subseteq W$, either $X \in u$ or $\overline{X} \in u$ (where $\overline{X}$ is the complement of X in W) $\triangleleft$

A collection $u_0 \subseteq \wp(W)$ has the **finite intersection property** provided for each $X, Y \in u_0$, $X \cap Y \neq \emptyset$.

Theorem 1.2 (Ultrafilter Theorem) *If a set $u_0 \subseteq \wp(W)$ has the finite intersection property, then u_0 can be extended to an ultrafilter over W (i.e., there is an ultrafilter u over W such that $u_0 \subseteq u$.*

Proof. Suppose that u_0 has the finite intersection property. Then, consider the set

$$u_1 = \{Z \mid \text{there are finitely many sets } X_1, \dots X_k \text{ such that } Z = X_1 \cap \cdots \cap X_k\}.$$

That is, u_1 is the set of finite intersections of sets from u_0. Note that $u_0 \subseteq u_1$, since u_0 has the finite intersection property, we have $\emptyset \notin u_1$, and by definition u_1 is closed under finite intersections. Now, define u_2 as follows:

$$u' = \{Y \mid \text{ there is a } Z \in u_1 \text{ such that } Z \subseteq Y\}$$

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We claim that u' is a consistent filter: $Y_1, Y_2 \in u'$ then there is a $Z_1 \in u_1$ such that $Z_1 \subseteq Y_1$ and $Z_2 \in u_1$ such that $Z_2 \subseteq Y_2$. Then, since $\emptyset \neq Z_1 \cap Z_2 \in u_1$, we have $Z_1 \cap Z_2 \subseteq Y_1 \cap Y_2$. Hence, $Y_1 \cap Y_2 \in u_1$. Also, if $X \in u_1$ then there is a $Z \in u_1$ such that $Z \subseteq X$. If $X \subseteq Y$, then $Z \subseteq Y$ and so $Y \in u_1$. Hence, u_1 is a consistent filter.

The next step is to show that u_1 can be extended to an ultrafilter. This follows almost directly from Zorn's Lemma[1]: Consider the set $\mathcal{Z}$ of all filters that extend u_1. That is, $\mathcal{Z} = \{v \mid u_1 \subseteq v \text{ and } v \text{ is a consistent filter}\}$. Note that $\mathcal{Z}$ is partially-ordered under the $\subseteq$-relation. Furthermore, the upper bound of any chain in $\mathcal{Z}$ (i.e., sequence of ultrafilters $v_0 \subseteq v_1 \subseteq \cdots$) is the union of all the filters in the chain. This collection of sets will be a consistent ultrafilter extending u_1, and so is contained in $\mathcal{Z}$. By Zorn's Lemma, $\mathcal{Z}$ must contain a maximal element. This maximal element must be an ultrafilter (containing u_1).
QED

Let $\mathcal{M} = \langle W, R, V\rangle$ be a Kripke model. Two functions are relevant to our analysis:

- $m : \wp(W) \to \wp(W)$ defined as
  $m(X) = \{w \mid \text{there is a } v \text{ such that } wRv \text{ and } v \in X\}$, and
- $l : \wp(W) \to \wp(W)$ defined as $l(X) = \{w \mid \text{ for all } v, \text{ if } wRv \text{ then } v \in X\}$.

Definition 1.3 (Ultrafilter Extension) An ultrafilter extension is a model $ue(\mathcal{M}) = \langle \mathsf{Uf}(W), R^{ue}, V^{ue}\rangle$ where

- $\mathsf{Uf}(W) = \{u \mid u \text{ is an ultrafilter over } W\}$,
- $uR^{ue}u'$ iff for all $X \subseteq W$, if $X \in u'$ then $m(X) \in u$, and
- $V^{ue}(p) = \{u \mid V(p) \in u\}$. ◁

Fact 1.4 *In an ultrafilter extension $ue(\mathcal{M}) = \langle \mathsf{Uf}(W), R^{ue}, V^{ue}\rangle$, we have $uR^{ue}u'$ iff $\{Y \mid l(Y) \in u\} \subseteq u'$.*

Proof. Left as an exercise. QED

Let $\mathcal{M} = \langle W, R, V\rangle$ be a model . The truth map $[\![\cdot]\!]_{\mathcal{M}} : \mathcal{L} \to \wp(W)$ is defined by induction on the structure of φ as follows:

- For $p \in \mathsf{At}$, $[\![p]\!]_{\mathcal{M}} = V(p)$

[1]Zorn's Lemma states that in a partially ordered set P, if every chain has an upper bound in P, then P contains at least on maximal element. The proof of Zorn's Lemma uses the Axiom of Choice (indeed, it is equivalent to the Axiom of Choice).

- $[\![\neg\varphi]\!]_{\mathcal{M}} = W - [\![\varphi]\!]_{\mathcal{M}}$
- $[\![\varphi \wedge \psi]\!]_{\mathcal{M}} = [\![\varphi]\!]_{\mathcal{M}} \cap [\![\psi]\!]_{\mathcal{M}}$
- $[\![\Diamond\varphi]\!]_{\mathcal{M}} = m([\![\varphi]\!]_{\mathcal{M}})$

Lemma 1.5 *For all models $\mathcal{M} = \langle W, R, V\rangle$, for all modal formulas φ, we have $[\![\varphi]\!]_{\mathcal{M}} \in u$ iff $ue(\mathcal{M}), u \models \varphi$.*

Proof. The proof is by induction on the structure of φ. Then we have,

Base case: φ is $p \in \mathsf{At}$.

$[\![p]\!]_{\mathcal{M}} \in u$	iff	$V(p) \in u$	(Definition of $[\![\cdot]\!]_{\mathcal{M}}$)
	iff	$u \in V^{ue}(p)$	(Definition of V^{ue})
	iff	$ue(\mathcal{M}), u \models p$	(Definition of truth in a model)

Induction Hypothesis: For all ψ less complex[2] than φ, $[\![\psi]\!]_{\mathcal{M}} \in u$ iff $ue(\mathcal{M}), u \models \psi$

Case 1: φ is $\neg\psi$:

$[\![\neg\psi]\!]_{\mathcal{M}} \in u$	iff	$W - [\![\psi]\!]_{\mathcal{M}} \in u$	(Definition of $[\![\cdot]\!]_{\mathcal{M}}$)
	iff	$[\![\psi]\!]_{\mathcal{M}} \notin u$	(Properties of an ultrafilter)
	iff	$ue(\mathcal{M}), u \not\models \psi$	(Induction Hypothesis)
	iff	$ue(\mathcal{M}), u \models \neg\psi$	(Definition of truth)

Case 2: φ is $\psi_1 \wedge \psi_2$

$[\![\psi_1 \wedge \psi_2]\!]_{\mathcal{M}} \in u$	iff	$[\![\psi_1]\!]_{\mathcal{M}} \cap [\![\psi_2]\!]_{\mathcal{M}} \in u$	(Definition of $[\![\cdot]\!]_{\mathcal{M}}$)
	iff	$[\![\psi_1]\!]_{\mathcal{M}} \in u$ and $[\![\psi_2]\!]_{\mathcal{M}} \in u$	(Properties of ultrafilters)
	iff	$ue(\mathcal{M}), u \models \psi_1$ and $ue(\mathcal{M}), u \models \psi_2$	(Induction hypothesis)
	iff	$ue(\mathcal{M}), u \models \psi_1 \wedge \psi_2$	(Definition of truth)

Case 3: φ is $\Diamond\psi$

Suppose that $ue(\mathcal{M}), u \models \Diamond\psi$. Then, there is a $u' \in \mathsf{Uf}(W)$ such that $uR^{ue}u'$ and $ue(\mathcal{M}), u' \models \psi$. By the induction hypothesis, $[\![\psi]\!]_{\mathcal{M}} \in u'$. By the definition of R^{ue}, we have $m([\![\psi]\!]_{\mathcal{M}}) \in u$, and so, $[\![\Diamond\psi]\!]_{\mathcal{M}} \in u$. Thus, we have shown that $ue(\mathcal{M}), u \models \Diamond\psi$ implies $[\![\Diamond\psi]\!]_{\mathcal{M}} \in u$.

Suppose that $[\![\Diamond\psi]\!]_{\mathcal{M}} \in u$. We must show $ue(\mathcal{M}), u \models \Diamond\psi$. Consider the set

$$u_0 = \{Y \mid l(Y) \in u\} \cup \{[\![\psi]\!]_{\mathcal{M}}\}$$

[2] Less complex means that ψ contains fewer connectives.

We claim that u_0 has the finite intersection property. We first show that $\{Y \mid l(Y) \in u\}$ is closed under finite intersections. It is enough to show that for any two sets $Y_1, Y_2 \in \{Y \mid l(Y) \in u\}$, $Y_1 \cap Y_2 \in \{Y \mid l(Y) \in u\}$ (why?). Suppose that $Y_1, Y_2 \in \{Y \mid l(Y) \in u\}$. Note that $l(Y_1 \cap Y_2) = l(Y_1) \cap l(Y_2)$. Then, since u is an ultrafilter and $l(Y_1), l(Y_2) \in u$, we have $l(Y_1 \cap Y_2) = l(Y_1) \cap l(Y_2) \in u$. Hence $Y_1 \cap Y_2 \in \{Y \mid l(Y) \in u\}$. Next we show that for any $Z \in \{Y \mid l(Y) \in u\}$, we have $Z \cap [\![\psi]\!]_{\mathcal{M}} \neq \emptyset$. Choose an arbitrary Z such that $l(Z) \in u$. We will show $Z \cap [\![\psi]\!]_{\mathcal{M}} \neq \emptyset$. Since $l(Z) \in u$ and $[\![\Diamond\psi]\!]_{\mathcal{M}} \in u$, we have $l(Z) \cap [\![\Diamond\psi]\!]_{\mathcal{M}} \in u$, and so $l(Z) \cap [\![\Diamond\psi]\!]_{\mathcal{M}} \neq \emptyset$. Let $w \in l(Z) \cap [\![\Diamond\psi]\!]_{\mathcal{M}}$. Then, there is a $v \in W$ such that $\mathcal{M}, v \models \psi$. I.e., wRv and $v \in [\![\psi]\!]_{\mathcal{M}}$. Since $w \in l(Y)$ and wRv, we have $v \in Y$. Hence, $v \in Y \cap [\![\psi]\!]_{\mathcal{M}}$. This implies that u_0 has the finite intersection property (why?).

By the ultrafilter theorem, there is an ultrafilter u' such that $u_0 \subseteq u'$. Since $\{Y \mid l(Y) \in u\} \subseteq u'$, we have $uR^{ue}u'$. By the induction hypothesis, since $[\![\psi]\!]_{\mathcal{M}} \in u'$, we have $ue(\mathcal{M}), u' \models \psi$. Hence, $ue(\mathcal{M}), u \models \Diamond\psi$. QED

Corollary 1.6 *For all models $\mathcal{M}$ and states w in $\mathcal{M}$, we have $w \leftrightsquigarrow u_w$, where u_w is the principle ultrafilter generated by w.*

Proof. Let $\mathcal{M} = \langle W, R, V\rangle$ be a model and $w \in W$. The principle ultrafilter generated by w is $u_w = \{X \subseteq W \mid w \in X\}$. Let φ be an arbitrary modal formula. We have $\mathcal{M}, w \models \varphi$ iff $w \in [\![\varphi]\!]_{\mathcal{M}}$ iff $[\![\varphi]\!]_{\mathcal{M}} \in u_w$ iff $ue(\mathcal{M}), u_w \models \varphi$ (the latter equivalence follows from the above Lemma). QED

Lemma 1.7 *For all models $\mathcal{M}$, $ue(\mathcal{M})$ is m-saturated.*

Proof. Suppose that $ue(\mathcal{M}) = \langle \mathsf{Uf}(W), R^{ue}, V^{ue}\rangle$ an ultrafilter extension of some model $\mathcal{M} = \langle W, R, V\rangle$. Let $u \in \mathsf{Uf}(W)$ be any state in $ue(\mathcal{M})$ and Σ be a arbitrary set of modal formulas. Suppose that every finite subset of Σ is satisfiable at some successor of u (i.e., for each finite set $\Delta \subseteq \Sigma$, there is a state $v_\Delta \in W$ such that $uR^{ue}v_\Delta$ and $ue(\mathcal{M}), v_\Delta \models \bigwedge\Delta$). We must find a state $v \in W$ such that $uR^{ue}v$ and $ue(\mathcal{M}), v \models \Sigma$ (i.e., for each $\psi \in \Sigma$, $ue(\mathcal{M}), v \models \psi$). Consider the set

$$v_0 = \{Y \mid l(Y) \in u\} \cup \{[\![\psi]\!]_{\mathcal{M}} \mid \psi \in \Sigma\}$$

We will show v_0 has the finite intersection property. Since $\{Y \mid l(Y) \in u\}$ is closed under finite intersections, it is enough to show that $Y \cap \bigcap\{[\![\psi]\!]_{\mathcal{M}} \mid \psi \in \Delta\} \neq \emptyset$ for some finite subset Δ of Σ. Note that $\bigcap\{[\![\psi]\!]_{\mathcal{M}} \mid \psi \in \Delta\} = [\![\bigwedge\Delta]\!]_{\mathcal{M}}$. Recall that Δ is satisfiable at some successor state v_Δ of u. That is, $uR^{ue}v_\Delta$ and $ue(\mathcal{M}), v_\Delta \models \bigwedge\Delta$. By Lemma 1.5, this means $[\![\bigwedge\Delta]\!]_{\mathcal{M}} \in v_\Delta$. By the definition of R^{ue}, we have $m([\![\bigwedge\Delta]\!]_{\mathcal{M}}) \in u$. Hence, $[\![\Diamond\bigwedge\Delta]\!]_{\mathcal{M}} \in u$. Since u is an ultrafilter, we have $l(Y) \cap [\![\Diamond \bigwedge \Delta]\!]_{\mathcal{M}} \in u$. Hence, (since $\emptyset \notin u$) there is a $x \in l(Y) \cap [\![\Diamond \bigwedge \Delta]\!]_{\mathcal{M}}$. This implies there is a y such that xRy and $y \in [\![\bigwedge \Delta]\!]_{\mathcal{M}}$. Since xRy and $x \in l(Y)$, we have $y \in Y$. This means that $y \in Y \cap [\![\bigwedge \Delta]\!]_{\mathcal{M}}$. Hence, v_0 has the finite intersection property.

By the ultrafilter theorem there is an ultrafilter v such that $v_0 \subseteq v$. By construction v is a successor u (i.e., $uR^{ue}v$) and by Lemma 1.5, we have for each $\psi \in \Sigma$, $ue(\mathcal{M}), v \models \psi$. Hence, Σ is satisfiable is some successor state of u. QED

Theorem 1.8 (Bisimulation Somewhere Else Theorem) *For all models $\mathcal{M}$ and $\mathcal{M}'$, we have $\mathcal{M}, w \leftrightsquigarrow \mathcal{M}', w'$ iff $ue(\mathcal{M}), u_w \underline{\leftrightarrow} ue(\mathcal{M}'), u_{w'}$, where u_w and $u_{w'}$ are the principle ultrafilters containing w and w' respectively.*

Proof. Suppose that $ue(\mathcal{M}), u_w \underline{\leftrightarrow} ue(\mathcal{M}'), u_{w'}$. Let φ be any model formula. We have

$$\begin{array}{llll} \mathcal{M}, w \models \varphi & \text{iff} & ue(\mathcal{M}), u_w \models \varphi & \text{(Corollary 1.6)} \\ & \text{iff} & ue(\mathcal{M}'), u_{w'} \models \varphi & \text{(Bisimulation implies modal equivalence)} \\ & \text{iff} & \mathcal{M}', w' \models \varphi & \text{(Corollary 1.6)} \end{array}$$

Suppose that $\mathcal{M}, w \leftrightsquigarrow \mathcal{M}', w'$. Then, by Corollary 1.6, we have $ue(\mathcal{M}), u_w \leftrightsquigarrow \mathcal{M}, w \leftrightsquigarrow \mathcal{M}', w' \leftrightsquigarrow ue(\mathcal{M}'), u_{w'}$. By Lemma 1.7, both $ue(\mathcal{M})$ and $ue(\mathcal{M}')$ are modally saturated. In modally saturated models, modal equivalence implies bisimularity. Hence, $ue(\mathcal{M}), u_w \underline{\leftrightarrow} ue(\mathcal{M}'), u_{w'}$. QED

2 The Standard Translation

Let $\mathcal{M} = \langle W, R, V \rangle$ be a Kripke model. The first-order language $\mathcal{L}_1$ is built from a signature containing unary predicate symbols Px corresponding to each $p \in \mathsf{At}$ and a binary predicate symbol Rxy. The standard translation is defined as follows:

Definition 2.1 (Standard Translation) The standard translation are functions $st_x : \mathcal{L} \to \mathcal{L}_1$ defined as follows:

$$\begin{array}{lcl} st_x(p) & = & Px \\ st_x(\neg\varphi) & = & \neg st_x(\varphi) \\ st_x(\varphi \wedge \psi) & = & st_x(\varphi) \wedge st_x(\psi) \\ st_x(\Box\varphi) & = & \forall y(Rxy \to st_y(\varphi)) \\ st_x(\Diamond\varphi) & = & \exists y(Rxy \wedge st_y(\varphi)) \end{array}$$

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Observation 2.2 *Modal logic falls in the two-variable fragment of* $\mathcal{L}_1$.

Proof. By carefully reusing bound variables, once can ensure that the translation of a modal formula uses only two variable. An example suffices to show how this works:

$$st_x(\Diamond\Diamond p) = \exists y(Rxy \wedge st_y(\Diamond p)) = \exists y(Rxy \wedge (\exists x(Ryx \wedge Px)))$$

QED

Lemma 2.3 *Let* $\mathcal{M} = \langle W, R, V\rangle$ *be a Kripke model. For each* $w \in W$, $\mathcal{M}, w \models \varphi$ *iff* $\mathcal{M} \Vdash st_x(\varphi)[x/w]$, *where* $\Vdash$ *denotes truth of* $\mathcal{L}_1$ *in a model* $\mathcal{M}$ *(viewed as a first-order structure).*

Proof. The simple but instructive proof is left to the reader. QED

Theorem 2.4 (Van Benthem Characterization Theorem) *A first-order formula* $\alpha(x)$ *(in the appropriate language) is invariant for bisimulation iff it is equivalent to the translation of a modal formula.*