Modal Logic
Preference Modal Logics

Eric Pacuit

University of Maryland, College Park
ai.stanford.edu/∼epacuit

May 1, 2012

Preference (Modal) Logics

x, y objects

$x \succeq y$: x is at least as good as y

Preference (Modal) Logics

x, y objects

$x \succeq y$: x is at least as good as y

1. $x \succeq y$ and $y \not\succeq x$ ($x \succ y$)
2. $x \not\succeq y$ and $y \succeq x$ ($y \succ x$)
3. $x \succeq y$ and $y \succeq x$ ($x \sim y$)
4. $x \not\succeq y$ and $y \not\succeq x$ ($x \perp y$)

Preference (Modal) Logics

x, y objects

$x \succeq y$: x is at least as good as y

1. $x \succeq y$ and $y \not\succeq x$ ($x \succ y$)
2. $x \not\succeq y$ and $y \succeq x$ ($y \succ x$)
3. $x \succeq y$ and $y \succeq x$ ($x \sim y$)
4. $x \not\succeq y$ and $y \not\succeq x$ ($x \perp y$)

Properties: transitivity, connectedness, etc.

Preference (Modal) Logics

Modal betterness model $\mathcal{M} = \langle W, \succeq, V \rangle$

Preference (Modal) Logics

Modal betterness model $\mathcal{M} = \langle W, \succeq, V \rangle$

Preference Modalities $\langle\succeq\rangle\varphi$: "there is a world at least as good (as the current world) satisfying φ"

$\mathcal{M}, w \models \langle\succeq\rangle\varphi$ iff there is a $v \succeq w$ such that $\mathcal{M}, v \models \varphi$

Preference (Modal) Logics

Modal betterness model $\mathcal{M} = \langle W, \succeq, V \rangle$

Preference Modalities $\langle \succeq \rangle \varphi$: "there is a world at least as good (as the current world) satisfying φ"

$\mathcal{M}, w \models \langle \succeq \rangle \varphi$ iff there is a $v \succeq w$ such that $\mathcal{M}, v \models \varphi$

$\mathcal{M}, w \models \langle \succ \rangle \varphi$ iff there is $v \succeq w$ and $w \not\succeq v$ such that $\mathcal{M}, v \models \varphi$

Preference (Modal) Logics

1. $\langle \succ \rangle \varphi \to \langle \succeq \rangle \varphi$
2. $\langle \succeq \rangle \langle \succ \rangle \varphi \to \langle \succ \rangle \varphi$
3. $\varphi \wedge \langle \succeq \rangle \psi \to (\langle \succ \rangle \psi \vee \langle \succeq \rangle (\psi \wedge \langle \succeq \rangle \varphi))$
4. $\langle \succ \rangle \langle \succeq \rangle \varphi \to \langle \succ \rangle \varphi$

Theorem The above logic (with Necessitation and Modus Ponens) is sound and complete with respect to the class of preference models.

J. van Benthem, O. Roy and P. Girard. *Everything else being equal: A modal logic approach to* ceteris paribus *preferences*. JPL, 2008.

Preference Modalities

$\varphi \geq \psi$: the state of affairs φ is at least as good as ψ (ceteris paribus)

G. von Wright. *The logic of preference*. Edinburgh University Press (1963).

Preference Modalities

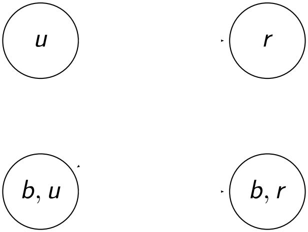
$\varphi \geq \psi$: the state of affairs φ is at least as good as ψ (ceteris paribus)

G. von Wright. *The logic of preference*. Edinburgh University Press (1963).

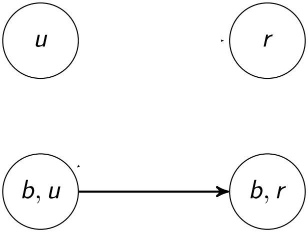
$\langle\Gamma\rangle^{\leq}\varphi$: φ is true in "better" world, *all things being equal*.

J. van Benthem, O. Roy and P. Girard. *Everything else being equal: A modal logic approach to* ceteris paribus *preferences*. JPL, 2008.

All Things Being Equal...

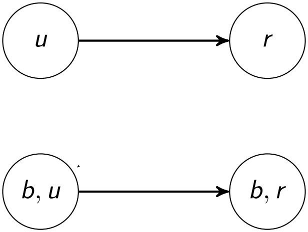


All Things Being Equal...



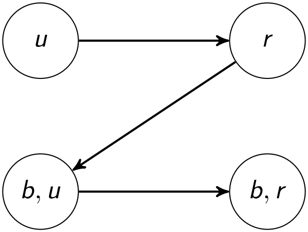
- With boots (b), I prefer my raincoat (r) over my umbrella (u)

All Things Being Equal...



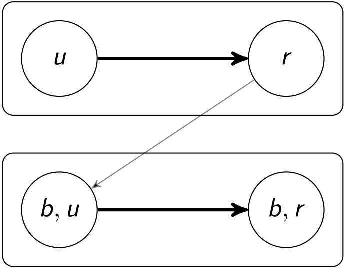
- With boots (b), I prefer my raincoat (r) over my umbrella (u)
- Without boots ($\neg b$), I also prefer my raincoat (r) over my umbrella (u)

All Things Being Equal...



- With boots (b), I prefer my raincoat (r) over my umbrella (u)
- Without boots ($\neg b$), I also prefer my raincoat (r) over my umbrella (u)
- But I do prefer an umbrella and boots over a raincoat and no boots

All Things Being Equal...



All things being equal, I prefer my raincoat over my umbrella

All Things Being Equal...

Let Γ be a set of (preference) formulas. Write $w \equiv_\Gamma v$ if for all $\varphi \in \Gamma$, $w \models \varphi$ iff $v \models \varphi$.

1. $\mathcal{M}, w \models \langle\Gamma\rangle\varphi$ iff there is a $v \in W$ such that $w \equiv_\Gamma v$ and $\mathcal{M}, v \models \varphi$.
2. $\mathcal{M}, w \models \langle\Gamma\rangle^{\leq}\varphi$ iff there is a $v \in W$ such that $w(\equiv_\Gamma \cap \leq)v$ and $\mathcal{M}, v \models \varphi$.
3. $\mathcal{M}, w \models \langle\Gamma\rangle^{<}\varphi$ iff there is a $v \in W$ such that $w(\equiv_\Gamma \cap <)v$ and $\mathcal{M}, v \models \varphi$.

All Things Being Equal...

Let Γ be a set of (preference) formulas. Write $w \equiv_\Gamma v$ if for all $\varphi \in \Gamma$, $w \models \varphi$ iff $v \models \varphi$.

1. $\mathcal{M}, w \models \langle\Gamma\rangle\varphi$ iff there is a $v \in W$ such that $w \equiv_\Gamma v$ and $\mathcal{M}, v \models \varphi$.
2. $\mathcal{M}, w \models \langle\Gamma\rangle^{\leq}\varphi$ iff there is a $v \in W$ such that $w(\equiv_\Gamma \cap \leq)v$ and $\mathcal{M}, v \models \varphi$.
3. $\mathcal{M}, w \models \langle\Gamma\rangle^{<}\varphi$ iff there is a $v \in W$ such that $w(\equiv_\Gamma \cap <)v$ and $\mathcal{M}, v \models \varphi$.

Key Principles:

- $\langle\Gamma'\rangle\varphi \rightarrow \langle\Gamma\rangle\varphi$ if $\Gamma \subseteq \Gamma'$
- $\pm\varphi \wedge \langle\Gamma\rangle(\alpha \wedge \pm\varphi) \rightarrow \langle\Gamma \cup \{\varphi\}\rangle\alpha$

All Things Being Equal...

Let Γ be a set of (preference) formulas. Write $w \equiv_\Gamma v$ if for all $\varphi \in \Gamma$, $w \models \varphi$ iff $v \models \varphi$.

1. $\mathcal{M}, w \models \langle\Gamma\rangle\varphi$ iff there is a $v \in W$ such that $w \equiv_\Gamma v$ and $\mathcal{M}, v \models \varphi$.
2. $\mathcal{M}, w \models \langle\Gamma\rangle^{\leq}\varphi$ iff there is a $v \in W$ such that $w(\equiv_\Gamma \cap \leq)v$ and $\mathcal{M}, v \models \varphi$.
3. $\mathcal{M}, w \models \langle\Gamma\rangle^{<}\varphi$ iff there is a $v \in W$ such that $w(\equiv_\Gamma \cap <)v$ and $\mathcal{M}, v \models \varphi$.

Key Principles:

- $\langle\Gamma'\rangle^{\leq}\varphi \rightarrow \langle\Gamma\rangle^{\leq}\varphi$ if $\Gamma \subseteq \Gamma'$
- $\pm\varphi \wedge \langle\Gamma\rangle^{\leq}(\alpha \wedge \pm\varphi) \rightarrow \langle\Gamma \cup \{\varphi\}\rangle^{\leq}\alpha$

Preference Lifting, I

Given a preference ordering $\preceq$ over a set of objects X, we want to **lift** this to an ordering $\hat{\preceq}$ over $\wp(X)$.

Given $\preceq$, what reasonable properties can we infer about $\hat{\preceq}$?

S. Barberá, W. Bossert, and P.K. Pattanaik. *Ranking sets of objects*. In Handbook of Utility Theory, volume 2. Kluwer Academic Publishers, 2004.

Preference Lifting, II

- You know that $x \prec y \prec z$
  Can you infer that $\{x, y\} \hat{\precsim} \{z\}$?

Preference Lifting, II

- You know that $x \prec y \prec z$
  Can you infer that $\{x, y\} \hat{\precsim} \{z\}$?

- You know that $x \prec y \prec z$
  Can you infer anything about $\{y\}$ and $\{x, z\}$?

Preference Lifting, II

- You know that $x \prec y \prec z$
  Can you infer that $\{x, y\} \hat{\prec} \{z\}$?

- You know that $x \prec y \prec z$
  Can you infer anything about $\{y\}$ and $\{x, z\}$?

- You know that $w \prec x \prec y \prec z$
  Can you infer that $\{w, x, y\} \hat{\preceq} \{w, y, z\}$?

Preference Lifting, II

- You know that $x \prec y \prec z$
  Can you infer that $\{x, y\} \hat{\prec} \{z\}$?

- You know that $x \prec y \prec z$
  Can you infer anything about $\{y\}$ and $\{x, z\}$?

- You know that $w \prec x \prec y \prec z$
  Can you infer that $\{w, x, y\} \hat{\preceq} \{w, y, z\}$?

- You know that $w \prec x \prec y \prec z$
  Can you infer that $\{w, x\} \hat{\prec} \{y, z\}$?

Preference Lifting, III

There are different interpretations of $X \mathrel{\hat{\preceq}} Y$:

- You will get one of the elements, but cannot control which.
- You can choose one of the elements.
- You will get the full set.

Preference Lifting, IV

Kelly Principle

(EXT) $\{x\} \hat{\precsim} \{y\}$ provided $x \prec y$

(MAX) $A \hat{\precsim} Max(A)$

(MIN) $Min(A) \hat{\precsim} A$

J.S. Kelly. *Strategy-Proofness and Social Choice Functions without Single-Valuedness*. Econometrica, 45(2), pp. 439 - 446, 1977.

Preference Lifting, IV

Gärdenfors Principle

(G1) $A \hat{\precsim} A \cup \{x\}$ if $a \prec x$ for all $a \in A$

(G2) $A \cup \{x\} \hat{\precsim} A$ if $x \prec a$ for all $a \in A$

P. Gärdenfors. *Manipulation of Social Choice Functions*. Journal of Economic Theory. 13:2, 217 - 228, 1976.

Preference Lifting, IV

Gärdenfors Principle

(G1) $A \mathrel{\hat{\precsim}} A \cup \{x\}$ if $a \prec x$ for all $a \in A$

(G2) $A \cup \{x\} \mathrel{\hat{\precsim}} A$ if $x \prec a$ for all $a \in A$

P. Gärdenfors. *Manipulation of Social Choice Functions*. Journal of Economic Theory. 13:2, 217 - 228, 1976.

Independence

(IND) $A \cup \{x\} \mathrel{\hat{\preceq}} B \cup \{x\}$ if $A \mathrel{\hat{\preceq}} B$ and $x \notin A \cup B$

Preference Lifting, V

Theorem (Kannai and Peleg). If $|X| \geq 6$, then no weak order satisfies both the Gärdenfors principle and independence.

Y. Kannai and B. Peleg. *A Note on the Extension of an Order on a Set to the Power Set*. Journal of Economic Theory, 32(1), pp. 172 - 175, 1984.

From Worlds to Sets, I

$\mathcal{M}, w \models \varphi \preceq_{\exists\exists} \psi$ iff there is s, t such that $\mathcal{M}, s \models \varphi$ and $\mathcal{M}, t \models \psi$ and $s \preceq t$

From Worlds to Sets, I

$\mathcal{M}, w \models \varphi \preceq_{\exists\exists} \psi$ iff there is s, t such that $\mathcal{M}, s \models \varphi$ and $\mathcal{M}, t \models \psi$ and $s \preceq t$

$\mathcal{M}, w \models \varphi \prec_{\exists\exists} \psi$ iff there is s, t such that $\mathcal{M}, s \models \varphi$ and $\mathcal{M}, t \models \psi$ and $s \prec t$

From Worlds to Sets, I

$\mathcal{M}, w \models \varphi \preceq_{\exists\exists} \psi$ iff there is s, t such that $\mathcal{M}, s \models \varphi$ and $\mathcal{M}, t \models \psi$ and $s \preceq t$

$\mathcal{M}, w \models \varphi \prec_{\exists\exists} \psi$ iff there is s, t such that $\mathcal{M}, s \models \varphi$ and $\mathcal{M}, t \models \psi$ and $s \prec t$

$\mathcal{M}, w \models \varphi \preceq_{\forall\exists} \psi$ iff for all s there is a t such that $\mathcal{M}, s \models \varphi$ implies $\mathcal{M}, t \models \psi$, and $s \preceq t$

From Worlds to Sets, I

$\mathcal{M}, w \models \varphi \preceq_{\exists\exists} \psi$ iff there is s, t such that $\mathcal{M}, s \models \varphi$ and $\mathcal{M}, t \models \psi$ and $s \preceq t$

$\mathcal{M}, w \models \varphi \prec_{\exists\exists} \psi$ iff there is s, t such that $\mathcal{M}, s \models \varphi$ and $\mathcal{M}, t \models \psi$ and $s \prec t$

$\mathcal{M}, w \models \varphi \preceq_{\forall\exists} \psi$ iff for all s there is a t such that $\mathcal{M}, s \models \varphi$ implies $\mathcal{M}, t \models \psi$, and $s \preceq t$

$\mathcal{M}, w \models \varphi \preceq_{\forall\exists} \psi$ iff for all s there is a t such that $\mathcal{M}, s \models \varphi$ implies $\mathcal{M}, t \models \psi$, and $s \prec t$

From Worlds to Sets, II

$$\varphi \preceq_{\exists\exists} \psi \ := \ E(\varphi \wedge \Diamond^{\preceq}\psi)$$

From Worlds to Sets, II

$$\varphi \preceq_{\exists\exists} \psi \; := \; \mathsf{E}(\varphi \wedge \Diamond^{\preceq}\psi)$$

$$\varphi \prec_{\exists\exists} \psi \; := \; \mathsf{E}(\varphi \wedge \Diamond^{\prec}\psi)$$

From Worlds to Sets, II

$$\varphi \preceq_{\exists\exists} \psi \;:=\; E(\varphi \wedge \Diamond^{\preceq}\psi)$$

$$\varphi \prec_{\exists\exists} \psi \;:=\; E(\varphi \wedge \Diamond^{\prec}\psi)$$

$$\varphi \preceq_{\forall\exists} \psi \;:=\; A(\varphi \rightarrow \Diamond^{\preceq}\psi)$$

From Worlds to Sets, II

$$\varphi \preceq_{\exists\exists} \psi \;:=\; E(\varphi \wedge \Diamond^{\preceq}\psi)$$

$$\varphi \prec_{\exists\exists} \psi \;:=\; E(\varphi \wedge \Diamond^{\prec}\psi)$$

$$\varphi \preceq_{\forall\exists} \psi \;:=\; A(\varphi \rightarrow \Diamond^{\preceq}\psi)$$

$$\varphi \prec_{\forall\exists} \psi \;:=\; A(\varphi \rightarrow \Diamond^{\prec}\psi)$$

From Worlds to Sets, III

$\mathcal{M}, w \models \varphi \preceq_{\forall\forall} \psi$ iff for all s, for all t, $\mathcal{M}, s \models \varphi$ and $\mathcal{M}, t \models \psi$ implies $s \preceq t$

From Worlds to Sets, III

$\mathcal{M}, w \models \varphi \preceq_{\forall\forall} \psi$ iff for all s, for all t, $\mathcal{M}, s \models \varphi$ and $\mathcal{M}, t \models \psi$ implies $s \preceq t$

$\mathcal{M}, w \models \varphi \prec_{\forall\forall} \psi$ iff for all s, for all t, $\mathcal{M}, s \models \varphi$ and $\mathcal{M}, t \models \psi$ implies $s \prec t$

From Worlds to Sets,IV

$$\varphi \preceq_{\forall\forall} \psi \;:=\; A(\psi \to \Box^{\preceq}\neg\varphi)$$

From Worlds to Sets,IV

$$\varphi \preceq_{\forall\forall} \psi \ := \ A(\psi \to \Box^{\preceq}\neg\varphi)$$

$$\varphi \prec_{\forall\forall} \psi \ := \ A(\psi \to \Box^{\prec}\neg\varphi)$$

From Worlds to Sets,IV

$$\varphi \preceq_{\forall\forall} \psi \;:=\; A(\psi \to \Box^{\preceq}\neg\varphi)$$

$$\varphi \prec_{\forall\forall} \psi \;:=\; A(\psi \to \Box^{\prec}\neg\varphi)$$

We must assume the ordering $\preceq$ *is total*

From Sets to Worlds

$$P_1 \gg P_2 \gg P_3 \gg \cdots \gg P_n$$

$x > y$ iff x and y differ in at least one P_i and the first P_i where this happens is one with $P_i x$ and $\neg P_i y$

F. Liu and D. De Jongh. *Optimality, belief and preference*. 2006.

Logics of Knowledge and Preference

$K(\varphi \succeq \psi)$: "Ann knows that φ is at least as good as ψ"

$K\varphi \succeq K\psi$: "knowing φ is at least as good as knowing ψ

Logics of Knowledge and Preference

$K(\varphi \succeq \psi)$: "Ann knows that φ is at least as good as ψ"

$K\varphi \succeq K\psi$: "knowing φ is at least as good as knowing ψ

$\mathcal{M} = \langle W, \sim, \succeq, V \rangle$

Logics of Knowledge and Preference

$K(\varphi \succeq \psi)$: "Ann knows that φ is at least as good as ψ"

$K\varphi \succeq K\psi$: "knowing φ is at least as good as knowing ψ

$\mathcal{M} = \langle W, \sim, \succeq, V \rangle$

J. van Eijck. *Yet more modal logics of preference change and belief revision*. manuscript, 2009.

F. Liu. *Changing for the Better: Preference Dynamics and Agent Diversity*. PhD thesis, ILLC, 2008.

$$A(\psi \to \langle \succeq \rangle \varphi) \quad \textit{vs.} \quad K(\psi \to \langle \succeq \rangle \varphi)$$

$$A(\psi \to \langle \succeq \rangle \varphi) \quad \textit{vs.} \quad K(\psi \to \langle \succeq \rangle \varphi)$$

Should preferences be restricted to information sets?

$$A(\psi \to \langle \succeq \rangle \varphi) \quad \textit{vs.} \quad K(\psi \to \langle \succeq \rangle \varphi)$$

Should preferences be restricted to information sets?

$\mathcal{M}, w \models \langle \succeq \cap \sim \rangle \varphi$ iff there is a v with $w \sim v$ and $w \preceq v$ such that $\mathcal{M}, v \models \varphi$

$K(\psi \to \langle \succeq \cap \sim \rangle \varphi)$

D. Osherson and S. Weinstein. *Preference based on reasons*. Review of Symbolic Logic, 2012.

$\varphi \succeq_X \psi$ "The agent considers φ at least as good as ψ for reason X"

$\varphi \succeq_X \psi$ "The agent considers φ at least as good as ψ for reason X"

i envisions a situation in which φ is true and that otherwise differs little from his actual situation. Likewise i envisions a world where ψ is true and otherwise differs little from his actual situation. Finally, there utility according to u_X of the first imagined situation exceeds that of the second.

p: "i purchases a fire alarm"

p: "i purchases a fire alarm"

$p \succ_1 \neg p$: u_1 measures safety

p: "i purchases a fire alarm"

$p \succ_1 \neg p$: u_1 measures safety

$p \prec_2 \neg p$: u_2 measures finances

p: "i purchases a fire alarm"

$p \succ_1 \neg p$: u_1 measures safety

$p \prec_2 \neg p$: u_2 measures finances

What is the status of $p \succ_{1,2} \neg p$? $p \prec_{1,2} \neg p$?

$(p \succ_1 \top) \succ_2 \top$: it's in your financial interest that your buying a low-power automobile is in you safety interesting — which might well be true inasmuch as low-power vehicles are cheaper.

$(p \succ_1 \top) \succ_2 \top$: it's in your financial interest that your buying a low-power automobile is in you safety interesting — which might well be true inasmuch as low-power vehicles are cheaper.

$\neg q \succ_1 (p \succ_2 q)$: from the point of view of family pride, you'd rather that your brother not run for mayor than that Miss Smith be the superior candidate.

At a set of atomic proposition, $\mathbb{S}$ a set of **reasons**.

$$\langle W, s, u, V \rangle$$

- W is a set of states
- $s : W \times \wp_{\neq\emptyset}(W) \to W$ is a selection function ($s(w, A) \in A$)
- $u : W \times \mathbb{S} \to \mathfrak{R}$ is a utility function
- $V : \mathrm{At} \to \wp(W)$ is a valuation function

At a set of atomic proposition, $\mathbb{S}$ a set of **reasons**.

$$\langle W, s, u, V \rangle$$

- W is a set of states
- $s : W \times \wp_{\neq\emptyset}(W) \to W$ is a selection function ($s(w, A) \in A$)
- $u : W \times \mathbb{S} \to \mathfrak{R}$ is a utility function
- $V : \mathrm{At} \to \wp(W)$ is a valuation function

$$\mathcal{M}, w \models \theta \succeq_X \psi \text{ iff } u_X(s(w, [\![\theta]\!]_{\mathcal{M}})) \geq u_X(s(w, [\![\psi]\!]_{\mathcal{M}}))$$

provided $[\![\theta]\!]_{\mathcal{M}} \neq \emptyset$ *and* $[\![\psi]\!]_{\mathcal{M}} \neq \emptyset$

$$\Diamond\varphi =_{\text{def}} \varphi \succeq_X \varphi$$
$$\Box\varphi =_{\text{def}} \neg(\neg\varphi \succeq_X \neg\varphi)$$

Reflexive: for all w if $w \in A$ then $s(w, A) = w$.

Reflexive: for all w if $w \in A$ then $s(w, A) = w$.

$\mathcal{M}$ is reflexive implies $(p \succeq_X \top) \vee (\neg p \succeq_X \top)$ is valid.

Reflexive: for all w if $w \in A$ then $s(w, A) = w$.

$\mathcal{M}$ is reflexive implies $(p \succeq_X \top) \vee (\neg p \succeq_X \top)$ is valid.

$$\Box(p \rightarrow (p \prec_X \neg p)) \wedge \Box(\neg p \rightarrow (\neg p \succ_X p))$$

Regular: if $A \subseteq B$ and $w_1 \in A$ then If $s(w, B) = w_1$ then $s(w, A) = w_1$.

Regular: if $A \subseteq B$ and $w_1 \in A$ then If $s(w, B) = w_1$ then $s(w, A) = w_1$.

$\mathcal{M}$ is regular implies $((p \vee q) \succ_X r) \rightarrow ((p \succ_X r) \vee (q \succ_X r))$ is valid.

$\mathcal{M}$ is regular and reflexive then
$((p \prec_1 \top) \succ_2 (q \prec_1 \top)) \rightarrow (\neg p \succ_2 \neg q)$ is valid.

"If it is ecologically better for p than for q to politically backfire the abstaining from p is ecologically better than abstaining from q. "

$\mathcal{M}$ is proximal if for all w and $A \neq \emptyset$, If $s(w, A) = w_1$ then there is no $w_2 \in A$ such that $V^{-1}(w) \Delta V^{-1}(w_2) \subset V^{-1}(w) \Delta V^{-1}(w_1)$, where Δ is the symmetric difference.

$\mathcal{M}$ is proximal if for all w and $A \neq \emptyset$, If $s(w, A) = w_1$ then there is no $w_2 \in A$ such that $V^{-1}(w) \Delta V^{-1}(w_2) \subset V^{-1}(w) \Delta V^{-1}(w_1)$, where Δ is the symmetric difference.

$(((p \wedge r) \succ_X (q \wedge r)) \wedge ((p \wedge \neg r) \succ_X (q \wedge \neg r))) \rightarrow (p \succ_X q)$ is invalid in the class of regular and in the class of proximal models, but valid in the class of models that are both proximal and regular.

$\mathcal{M}$ is proximal if for all w and $A \neq \emptyset$, If $s(w, A) = w_1$ then there is no $w_2 \in A$ such that $V^{-1}(w) \Delta V^{-1}(w_2) \subset V^{-1}(w) \Delta V^{-1}(w_1)$, where Δ is the symmetric difference.

$(((p \wedge r) \succ_X (q \wedge r)) \wedge ((p \wedge \neg r) \succ_X (q \wedge \neg r))) \rightarrow (p \succ_X q)$ is invalid in the class of regular and in the class of proximal models, but valid in the class of models that are both proximal and regular.

$(p \wedge ((p \wedge q) \succ_X r)) \rightarrow (q \succ_X r)$

Thank you!