Lecture 2: Model Theoretic Constructions in Modal Logic

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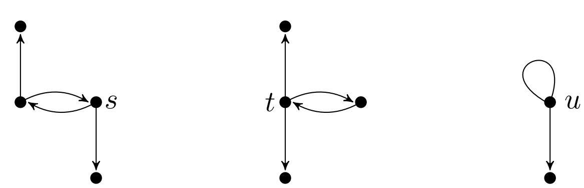
1 Basic Propositional Modal Logic

- **Language**: $p \mid \neg\varphi \mid \varphi \vee \psi \mid \Diamond\psi$, $p \in \mathsf{At}$ (atomic propositions), Boolean connectives defined as usual, $\Box\varphi \ := \ \neg\Diamond\neg\varphi$
- **Frame**: $\langle W, R\rangle$, where $W \neq \emptyset$ and $R \subseteq W \times W$
- **Model**: $\langle W, R, V\rangle$, where $\langle W, R\rangle$ is a frame and $V : \mathsf{At} \to \wp(W)$ (Kripke structure)
- **Truth at a state in a model**: $\mathcal{M}, w \models \varphi$
  - $\mathcal{M}, w \models p$ iff $w \in V(p)$
  - $\mathcal{M}, w \models \neg\varphi$ iff $\mathcal{M}, w \not\models \varphi$
  - $\mathcal{M}, w \models \varphi \wedge \psi$ iff $\mathcal{M}, w \models \varphi$ and $\mathcal{M}, w \models \psi$
  - $\mathcal{M}, w \models \Box\varphi$ iff for all $v \in W$, if wRv then $\mathcal{M}, v \models \varphi$
  - $\mathcal{M}, w \models \Diamond\varphi$ iff there is a $v \in W$ such that wRv and $\mathcal{M}, v \models \varphi$
- **Local vs. global truth**: True at a state, valid in a model (true at all states), locally valid in a frame (true at a state in all models based on the frame), valid in a frame (valid in all models based on the frame), etc.

2 Basic Model Theory

Which pair of states cannot be distinguished by a modal formula?



$\mathcal{K}$ $\qquad$ $\mathcal{M}$ $\qquad$ $\mathcal{N}$

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- $PKS(\varphi) = \{(\mathcal{M}, w) \mid \mathcal{M}, w \models \varphi\}$
- $KS(\varphi) = \{\mathcal{M} \mid \mathcal{M} \models \varphi$ for all $w \in dom(\mathcal{M})\}$
- $PFR(\varphi) = \{(\mathcal{F}, w) \mid (\mathcal{F}, V), w \models \varphi$ for all valuations $V\}$
- $FR(\varphi) = \{\mathcal{F} \mid (\mathcal{F}, V), w \models \varphi$ for all $w \in dom(\mathcal{M})$ and valuations $V\}$

<u>**Model Constructions**</u>

- **Disjoint Union**: Let $\mathcal{M}_1 = \langle W_1, R_1, V_1 \rangle$ and $\mathcal{M}_2 = \langle W_2, R_2, V_2 \rangle$. The disjoint union is the structure $\mathcal{M}_1 \uplus \mathcal{M}_2 = \langle W, R, V \rangle$ where
  - $W = W_1 \cup W_2$
  - $R = R_1 \cup R_2$
  - for all $p \in \mathsf{At}$, $V(p) = V_1(p) \cup V_2(p)$

  Lemma For each collection of Kripke structures $\{\mathcal{M}_i \mid i \in I\}$, for each $w \in W_i$, $\mathcal{M}_i, w \models \varphi$ iff $\biguplus_{i \in I} \mathcal{M}_i, w \models \varphi$
  Fact The universal modality is not definable in the basic modal language.

- **Generated Submodel**: $\mathcal{M}' = \langle W', R', V' \rangle$ is a generated submodel of $\mathcal{M} = \langle W, R, V \rangle$ provided
  - $W' \subseteq W$ is R-closed:
    for each $w' \in W$ and $v \in W$, if wRv then $v \in W'$.
  - $R' = R \cap W' \times W'$
  - for all $p \in \mathsf{At}$, $V'(p) = V(p) \cap W'$

  Lemma If $\mathcal{M}'$ is a generated submodel of $\mathcal{M}$ then for each $w \in W'$, $\mathcal{M}', w \models \varphi$ iff $\mathcal{M}, w \models \varphi$
  Fact The backwards looking modality is not definable in the basic modal language.

- **Bounded Morphism** A bounded morphism between models $\mathcal{M} = \langle W, R, V \rangle$ and $\mathcal{M}' = \langle W', R', V' \rangle$ is a function f with domain W and range W' such that:

  Atomic harmony: for each $p \in \mathsf{At}$, $w \in V(p)$ iff $f(w) \in V'(p)$
  Morphism: if wRv then $f(w)Rf(v)$
  Zag: if $f(w)R'v'$ then $\exists v \in W$ such that $f(v) = v'$ and wRv

  Lemma If $\mathcal{M}'$ is a bounded morphic image of $\mathcal{M}$ then for each $w \in W$, $\mathcal{M}, w \models \varphi$ iff $\mathcal{M}', f(w) \models \varphi$
  Fact Counting modalities are not definable in the basic modal language (eg., $\Diamond_1 \varphi$ iff φ is true in more than 1 accessible world).

- **Tree Unfoldings**: The unfolding of $\mathcal{M} = \langle W, R, V \rangle$ with root w is $\vec{\mathcal{M}} = \langle \vec{W}, \vec{R}, \vec{V} \rangle$, where $\vec{W}$ is the set of paths starting at w, $(w, \ldots, w_n) \vec{R} (w, \ldots, w_n, w_{n+1})$ iff $w_n R w_{n+1}$ and $(w, \ldots, w_n) \in V(p)$ iff $w_n \in V(p)$.

  Lemma. Every satisfiable modal formula is satisfiable at the root of a tree.

- **Bisimulation**: A bisimulation between $\mathcal{M} = \langle W, R, V \rangle$and $\mathcal{M}' = \langle W', R', V' \rangle$ is a non-empty binary relation $Z \subseteq W \times W'$ such that whenever wZw':

Atomic harmony: for each $p \in \mathsf{At}$, $w \in V(p)$ iff $w' \in V'(p)$
Zig: if wRv, then $\exists v' \in W'$ such that vZv' and $w'R'v'$
Zag: if $w'R'v'$ then $\exists v \in W$ such that vZv' and wRv

- *We write* $\mathcal{M}, w \underline{\leftrightarrow} \mathcal{M}', w'$ *if there is a* Z *such that* wZw'.
- *We write* $\mathcal{M}, w \leftrightsquigarrow \mathcal{M}', w'$ *iff* $\forall \varphi \in \mathcal{L}$, $\mathcal{M}, w \models \varphi$ *iff* $\mathcal{M}', w' \models \varphi$.
- **Lemma** If $\mathcal{M}, w \underline{\leftrightarrow} \mathcal{M}', w'$ then $\mathcal{M}, w \leftrightsquigarrow \mathcal{M}', w'$.
- **Lemma** On finite frames, if $\mathcal{M}, w \leftrightsquigarrow \mathcal{M}', w'$ then $\mathcal{M}, w \underline{\leftrightarrow} \mathcal{M}', w'$.
- **Lemma** On m**-saturated frames**, if $\mathcal{M}, w \leftrightsquigarrow \mathcal{M}', w'$ then $\mathcal{M}, w \underline{\leftrightarrow} \mathcal{M}', w'$.

Proposition. Any Kripke structure is the bounded morphic image of a disjoint union of rooted Kripke structures (in fact, tree structures).

Fact. Closure under generated subframe, bounded morphic images, and disjoint unions is not sufficient to guarantee definability by a modal formula for a class of frames. (eg., frames defined by $\forall x \exists y(xRy \wedge yRy)$).

- **Ultrafilter Extensions**: Let $m(X) = \{w \mid \text{there is a } v \text{ such that } wRv \text{ and } v \in X\}$ and $l(X) = \overline{m(\overline{X})} = \{w \mid \text{ for all } v, \text{ if } wRv \text{ then } v \in X\}$. An ultrafilter extension is a model

$$ue(\mathcal{M}) = \langle Uf(W), R^{ue}, V^{ue} \rangle$$

where $Uf(W) = \{u \mid u \text{ is an ultrafilter over } W\}$, $uR^{ue}u'$ iff for all $X \subseteq W$, if $X \in u'$ then $m(X) \in u$, and $V(p) = \{u \mid V(p) \in u\}$.

Fact. For all models $\mathcal{M}$, $w \leftrightsquigarrow u_w$, where u_w is the principle ultrafilter generated by w.
Fact. For all models $\mathcal{M}$, $ue(\mathcal{M})$ is m-saturated.
Fact. $\mathcal{M}, w \leftrightsquigarrow \mathcal{M}', w'$ iff $ue(\mathcal{M}), u_w \underline{\leftrightarrow} ue(\mathcal{M}'), u_{w'}$