Modal Logic
Introductory Lecture

Eric Pacuit

University of Maryland, College Park
ai.stanford.edu/∼epacuit

January 31, 2012

Setting the Stage

Modern Modal Logic began with C.I. Lewis' dissatisfaction with the material conditional ($\rightarrow$).

Dorothy Edgington's Proof of the Existence of God

X	Y	$X \rightarrow Y$
T	T	T
T	F	F
F	T	T
F	F	T

$$\neg G \rightarrow \neg(P \rightarrow A)$$

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$$\neg G \to \underset{\neg P}{\overbrace{\neg(P \to A)}^{F}}$$

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$$\neg P$$

$$\overline{\quad G \quad}$$

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God exists!

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Judge: $\neg(G \to A)$ $\Leftrightarrow G \wedge \neg A$, therefore G!

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Gradually, the study of the modalities themselves became dominant, with the study of "implication" developing into a separate topic.

Setting the Stage

$\Box\varphi$: "It is *necessarily* the case that φ" ("It must be that φ")

$\Diamond\varphi$: "It is *possible* that φ" ("It can/might be that φ")

Setting the Stage: Different Senses of "Possibility"

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- You may borrow but you may not steal. ("morally acceptable")
- It might rain tomorrow ("epistemic possibility")

The History of Modal Logic

R. Goldblatt. *Mathematical Modal Logic: A View of its Evolution*. Handbook of the History of Logic, Vol. 7, 2006.

P. Balckburn, M. de Rijke, and Y. Venema. *Modal Logic*. Section 1.7, Cambridge University Press, 2001.

R. Ballarin. *Modern Origins of Modal Logic*. Stanford Encyclopedia of Philosophy, 2010.

What is a modal?

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tense: henceforth, eventually, hitherto, deviously, now, tomorrow, yesterday, since, until, inevitably, finally, ultimately, endlessly, it will have been, it is being,. . .

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dynamic: after the program/computation/action finishes, the program enables, throughout the computation

geometric: it is locally the case that

metalogic: it is valid/satisfiable/provable/consistent that

The Basic Modal Language

A formula of Modal Logic is defined *inductively*:

1. Any atomic propositional variable is a formula
2. If P and Q are formula, then so are $\neg P$, $P \wedge Q$, $P \vee Q$ and $P \rightarrow Q$
3. If P is a formula, then so is $\Box P$ and $\Diamond P$

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Boolean Logic

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Unary operator

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Eg., $\Box(P \rightarrow \Diamond Q) \vee \Box\Diamond\neg R$

Modal Formulas: $\neg\Box\varphi \to \psi$

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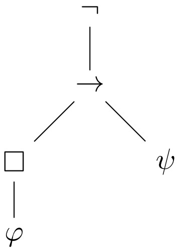

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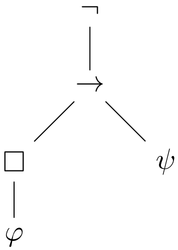
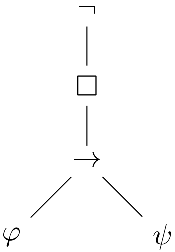


$\neg\Box(\varphi \to \psi)$



$(\neg\Box\varphi \to \psi)$

Modal Formulas: $\neg\Box\varphi \rightarrow \psi$

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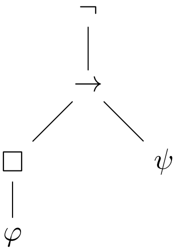

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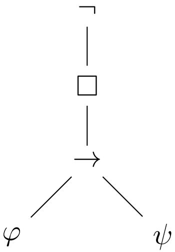

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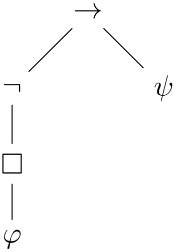

Narrow vs. Wide Scope

"If you do p, you must also do q"

- $p \to \Box q$
- $\Box(p \to q)$

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"If Bob is a bachelor, then he is necessarily unmarried"

- $B \to \Box U$
- $\Box(B \to U)$

de dicto vs. de re

"I know that someone appreciates me"

- $\Box\exists x A(x, e)$ (*de dicto*)
- $\exists x \Box A(x, e)$ (*de re*)

Iterations of Modal Operators

$\Box\varphi \to \Box\Box\varphi$: If I know, do I know that I know?

$\neg\Box\varphi \to \Box\neg\Box\varphi$: If I don't know, do I know that I don't know?

- Modal reasoning patterns
- Formal modeling

Deontic Logic

OA means A is obligatory
PA means A is permitted

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Is the following argument valid?

$$\frac{\text{If } A \text{ then } B \ (A \to B)}{\text{If } A \text{ is obligatory then so is } B \ (OA \to OB)}$$

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1. Jones murders Smith. (M)
2. If Jones murders Smith, then Jones ought to murder Smith gently. ($M \to OG$)

(first discussed by J. Forrester in 1984)

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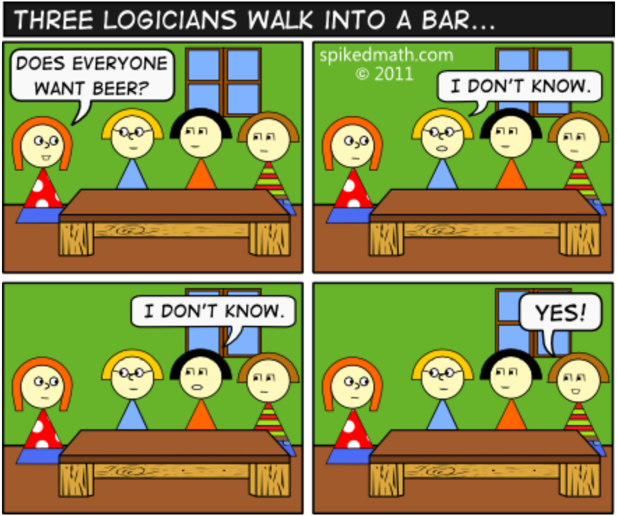
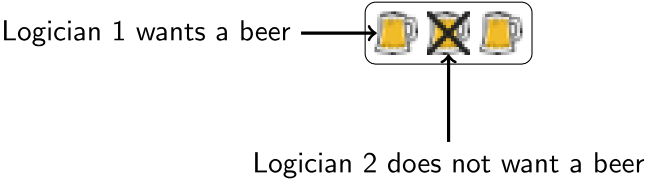

THREE LOGICIANS WALK INTO A BAR...
DOES EVERYONE WANT BEER?
spikedmath.com
© 2011
I DON'T KNOW.
I DON'T KNOW.
YES!

States

States

Logician 1 wants a beer

States



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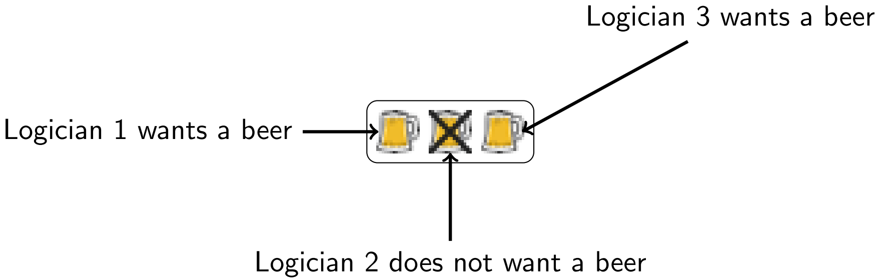
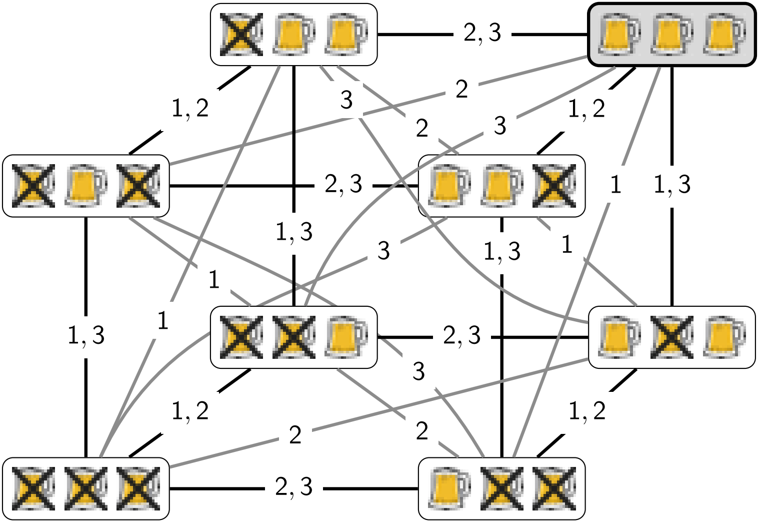

Logician 3 wants a beer
Logician 1 wants a beer
Logician 2 does not want a beer

A Model of the Logicians' Information



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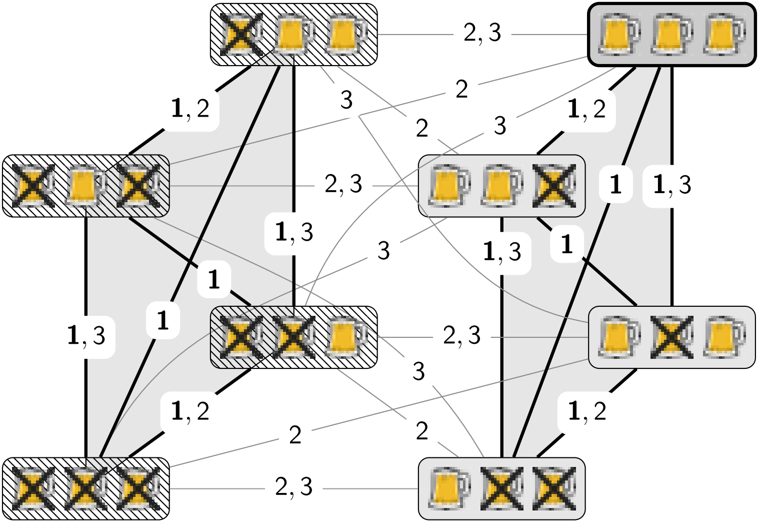

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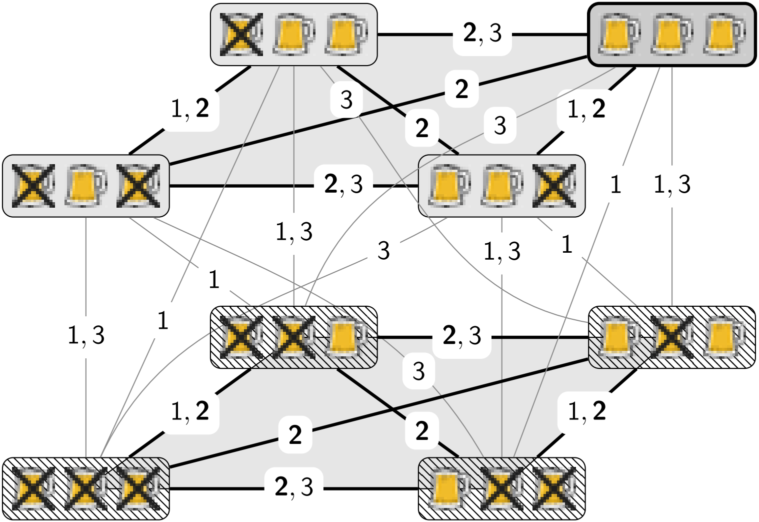

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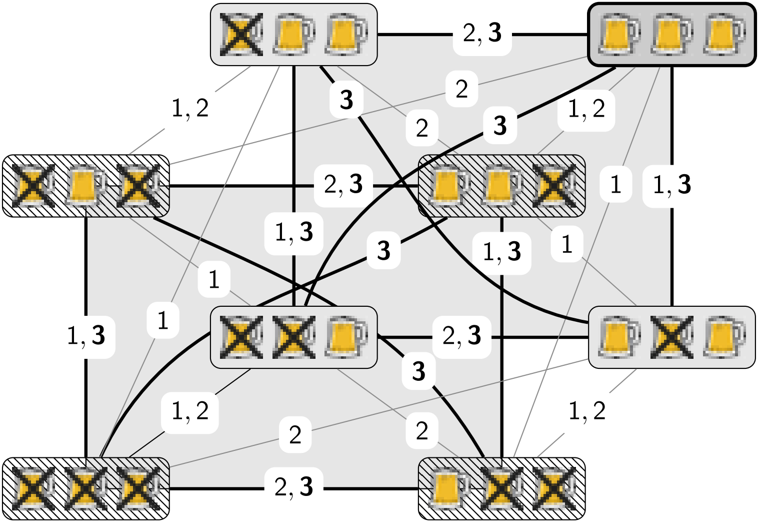

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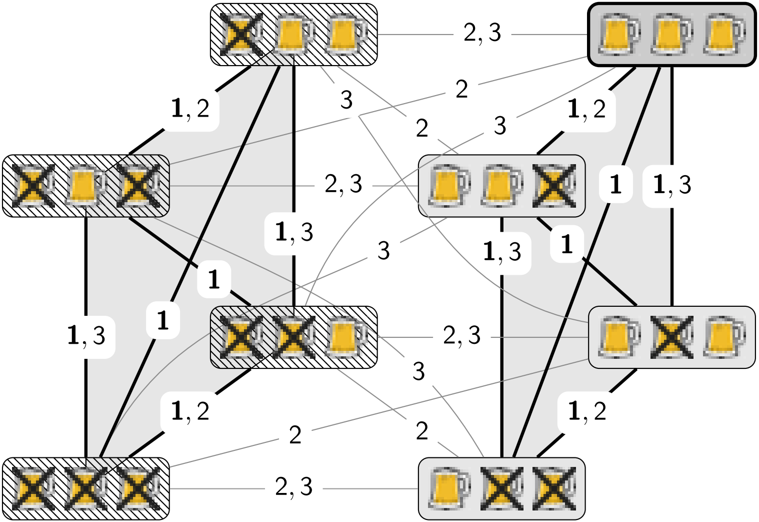

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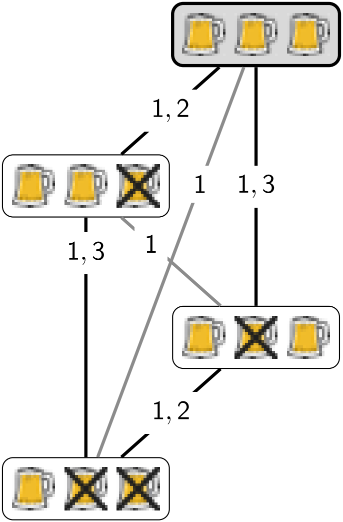
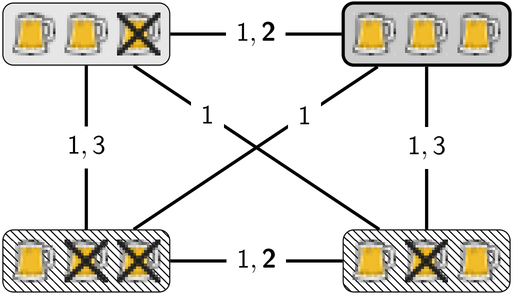


Logician 2: “I don’t know”



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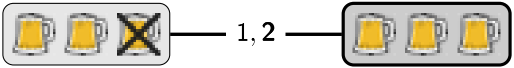
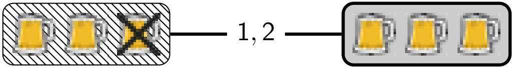


Logician 3: “Yes!”



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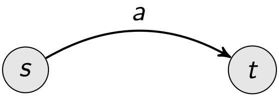

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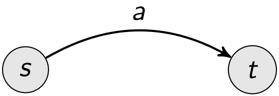
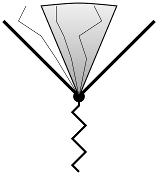
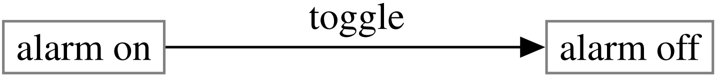


2. Actions *restrict* the set of possible future histories.

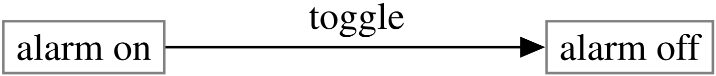


J. van Benthem, H. van Ditmarsch, J. van Eijck and J. Jaspers. *Chapter 6: Propositional Dynamic Logic*. Logic in Action Online Course Project, 2011.

Examples



Examples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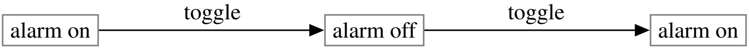
toggle
alarm off


alarm on
toggle
alarm off
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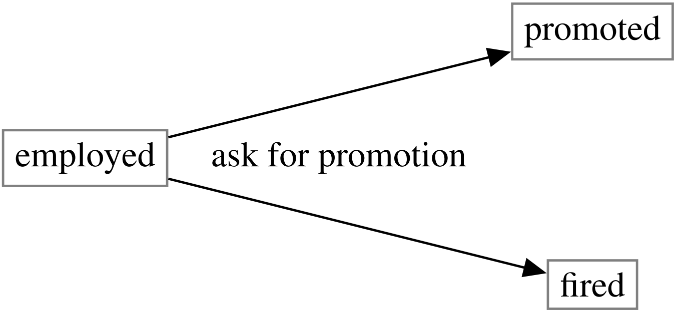
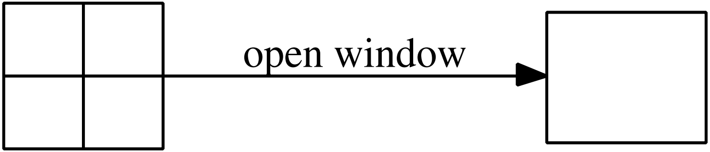
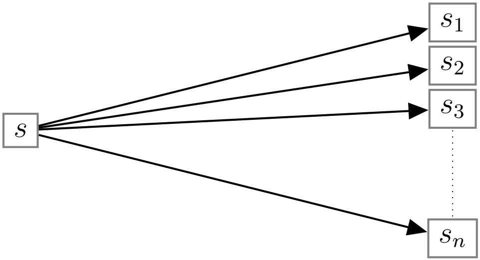
Examples



Examples




s
s_1
s_2
s_3
s_n

Semantics for Propositional Modal Logic

1. Relational semantics (i.e., Kripke semantics)
2. Algebraic semantics (BAO: Boolean algebras with operators)
3. Topological semantics (Closure algebras)
4. Category-theoretic (Coalgebras)

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Some Warm-up Questions

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Kripke Structures

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The main idea:

- ‘It is sunny outside’ is currently true

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- ‘It is sunny outside’ is currently true, but it is not necessary (for example, if we were currently in Amsterdam).

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- We say φ is **necessary** provided φ is true in all (relevant) situations (states, worlds, possibilities).

Kripke Structures

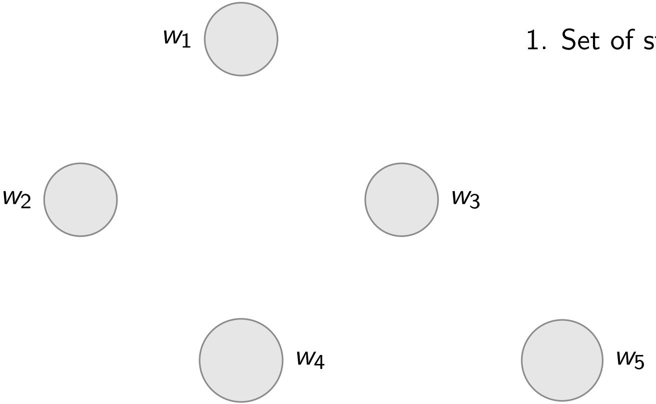
The main idea:

- ‘It is sunny outside’ is currently true, but it is not necessary (for example, if we were currently in Amsterdam).

- We say φ is **necessary** provided φ is true in all (relevant) situations (states, worlds, possibilities).

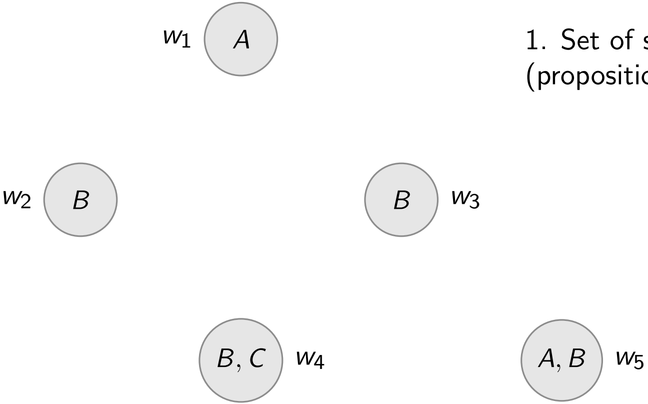
- A **Kripke structure** is
  1. A set of states, or worlds (each world specifies the truth value of all propositional variables)
  2. A **relation** on the set of states (specifying the “relevant situations”)

A Kripke Structure



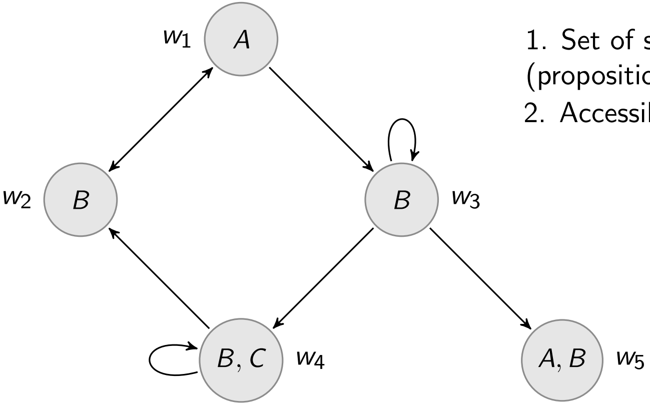
1. Set of states

A Kripke Structure



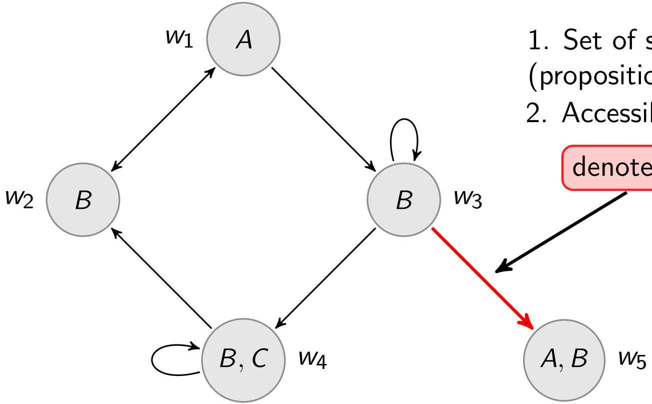
1. Set of states (propositional valuations)

A Kripke Structure



1. Set of states (propositional valuations)
2. Accessibility relation

A Kripke Structure



1. Set of states
(propositional valuations)
2. Accessibility relation

denoted $w_3 R w_5$

Truth of Modal Formulas

Model: $\mathcal{M} = \langle W, R, V \rangle$ where $W \neq \emptyset$, $R \subseteq W \times W$ and $V : \text{At} \to \wp(W)$ (At is the set of atomic propositions).

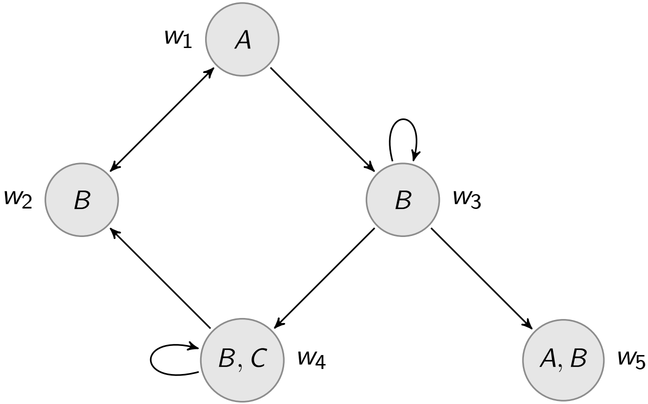
Truth of Modal Formulas

Model: $\mathcal{M} = \langle W, R, V \rangle$ where $W \neq \emptyset$, $R \subseteq W \times W$ and $V : \text{At} \to \wp(W)$ (At is the set of atomic propositions).

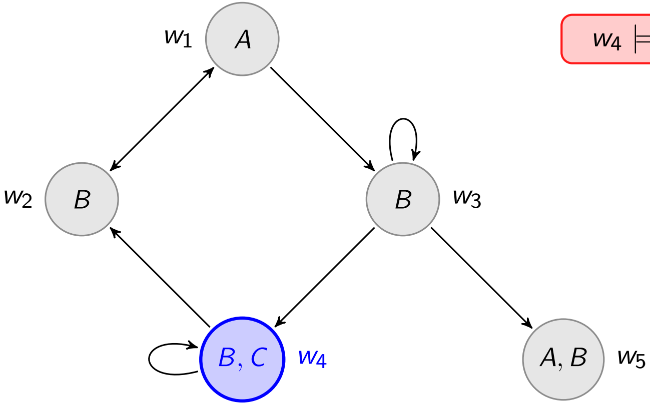
Truth at a state in a model: $\mathcal{M}, w \models \varphi$

- $\mathcal{M}, w \models p$ iff $w \in V(p)$
- $\mathcal{M}, w \models \neg\varphi$ iff $\mathcal{M}, w \not\models \varphi$
- $\mathcal{M}, w \models \varphi \wedge \psi$ iff $\mathcal{M}, w \models \varphi$ and $\mathcal{M}, w \models \psi$
- $\mathcal{M}, w \models \Box\varphi$ iff for all $v \in W$, if wRv then $\mathcal{M}, v \models \varphi$
- $\mathcal{M}, w \models \Diamond\varphi$ iff there is a $v \in W$ such that $\mathcal{M}, v \models \varphi$

Example

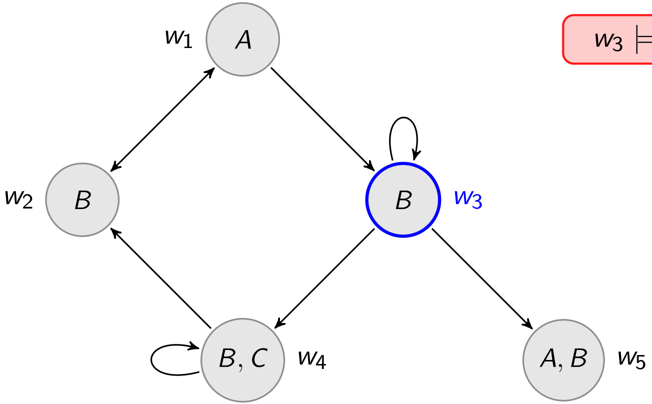


Example



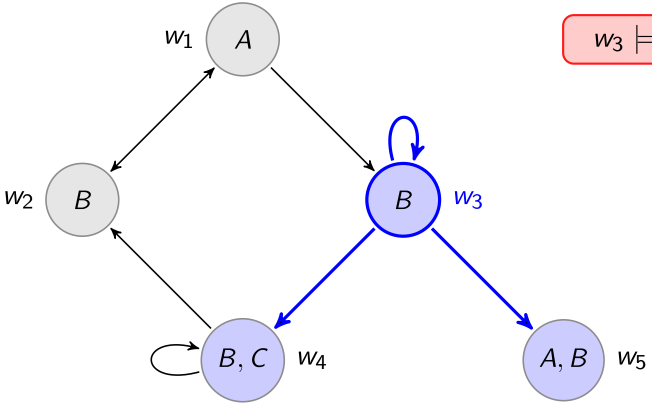
$w_4 \models B \wedge C$

Example



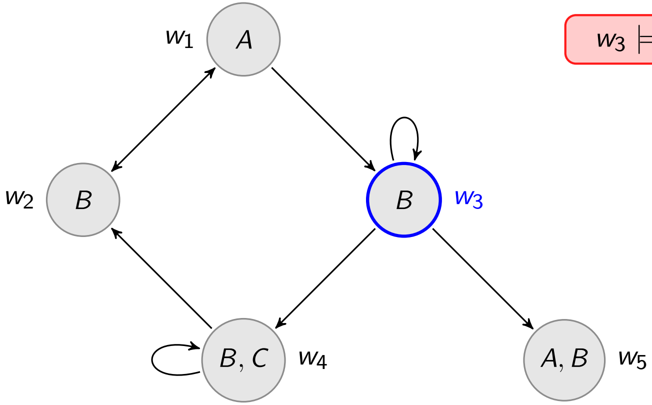
$w_3 \models \Box B$

Example



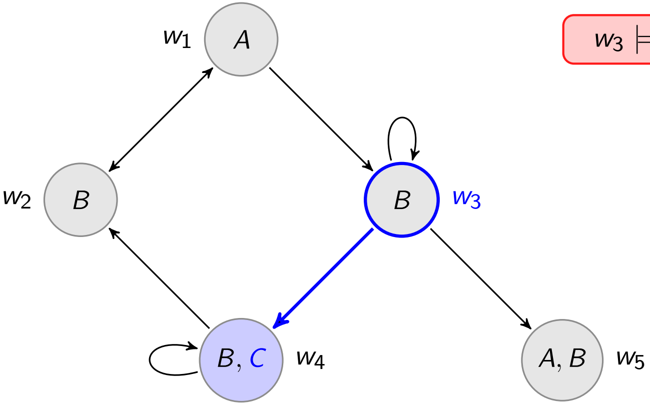
$w_3 \models \Box B$

Example



$w_3 \models \Diamond C$

Example



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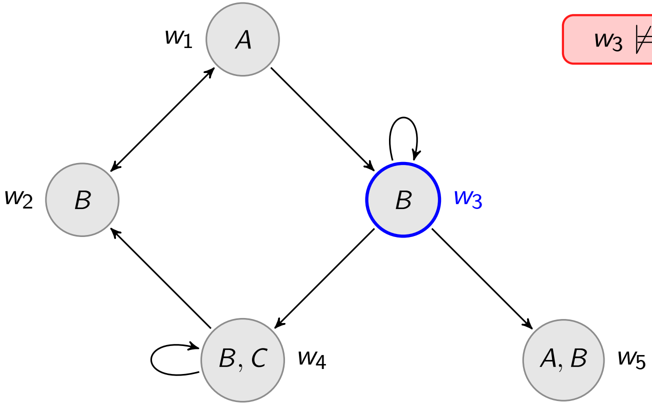
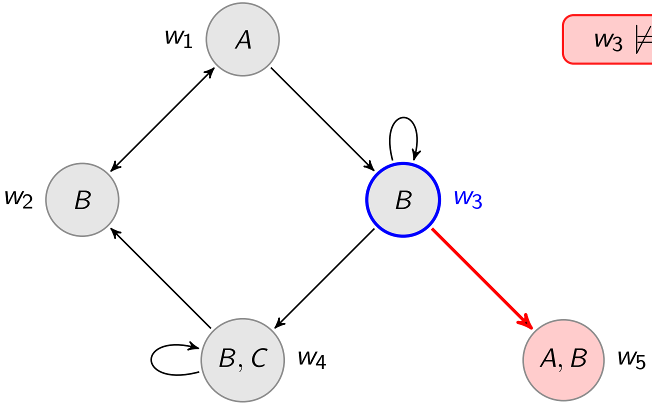


$w_3 \not\models \Box C$

Example



$w_3 \not\models \Box C$

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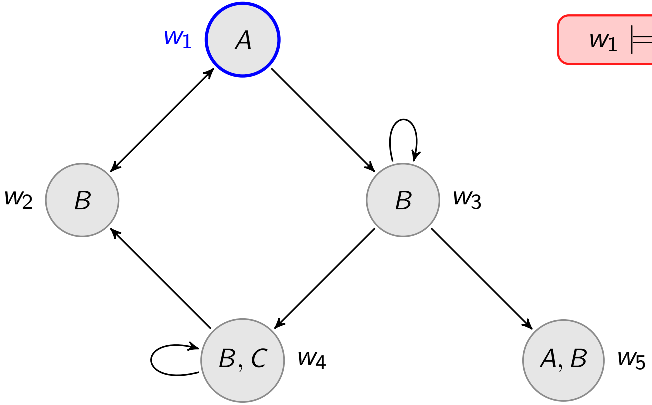
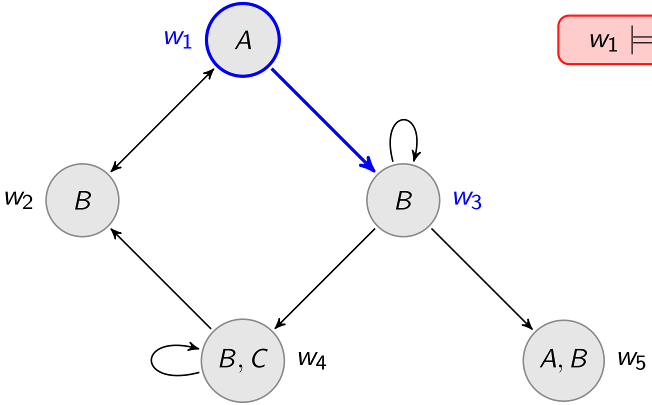


$w_1 \models \Diamond\Box B$

Example



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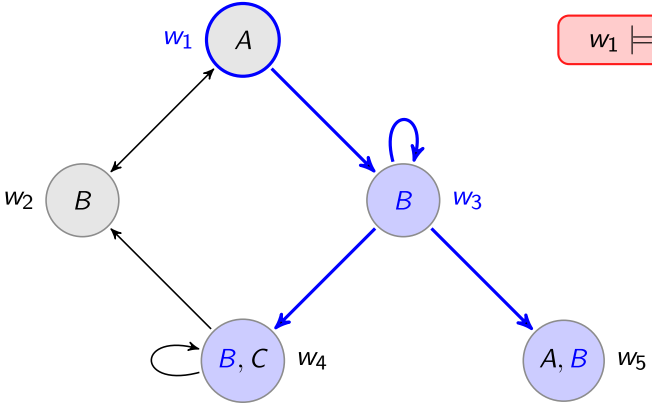
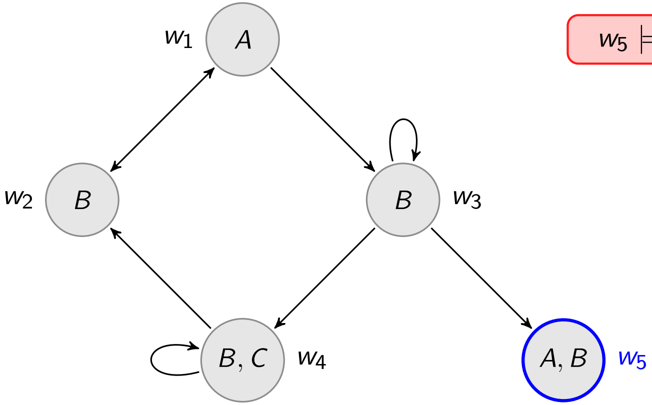
Example



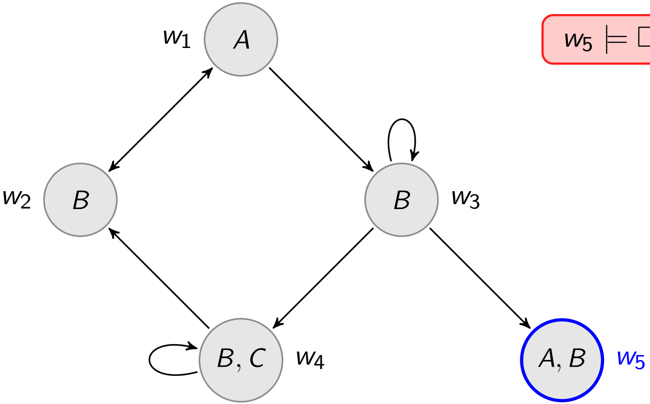
$w_1 \models \Diamond\Box B$

Example



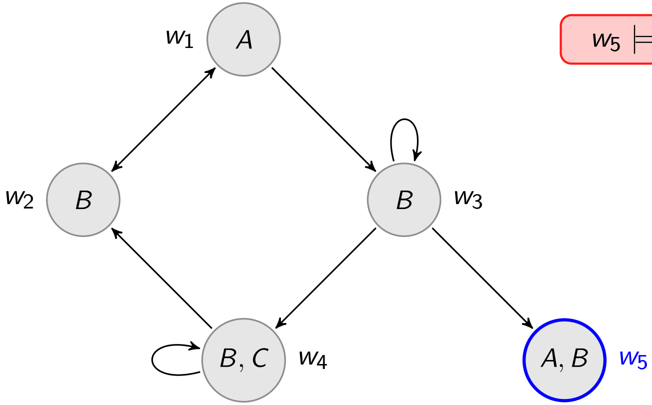
$w_5 \models \Box C$

Example

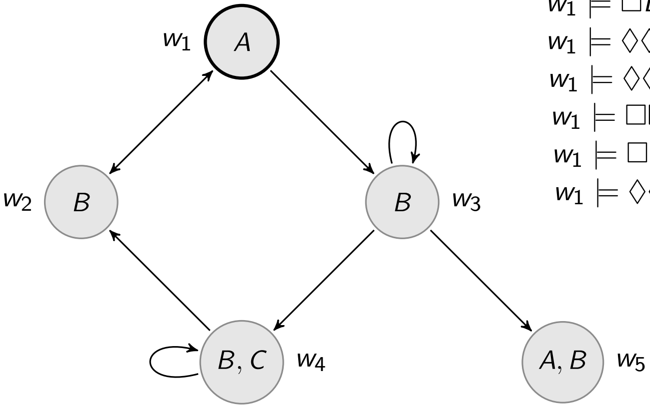


$w_5 \models \Box(B \wedge \neg B)$

Example



$w_5 \models \neg\Diamond B$



$w_1 \models \Box B \wedge B$?

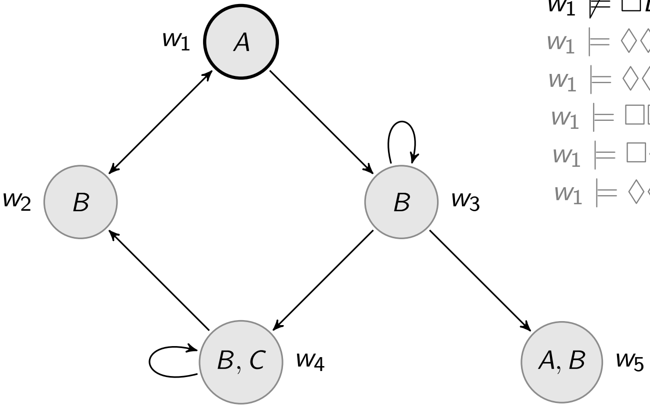
$w_1 \models \Diamond\Diamond B$?

$w_1 \models \Diamond\Diamond\Diamond B$?

$w_1 \models \Box\Box B$?

$w_1 \models \Box\Diamond C$?

$w_1 \models \Diamond\Diamond C$?



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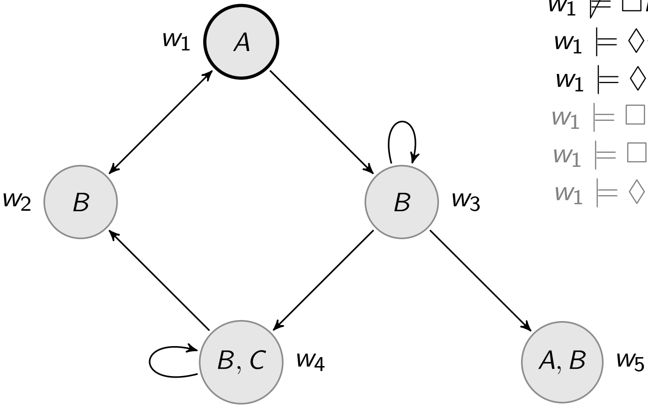

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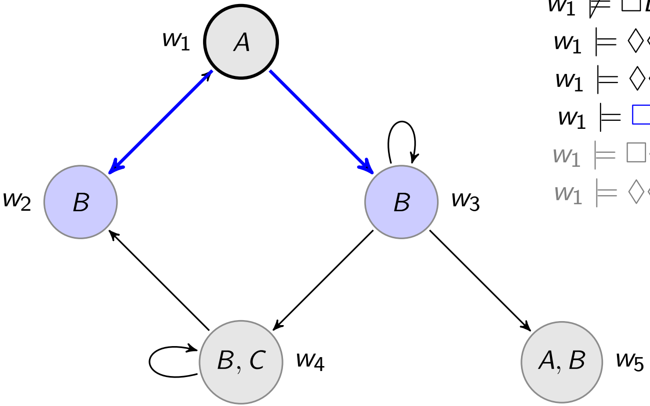

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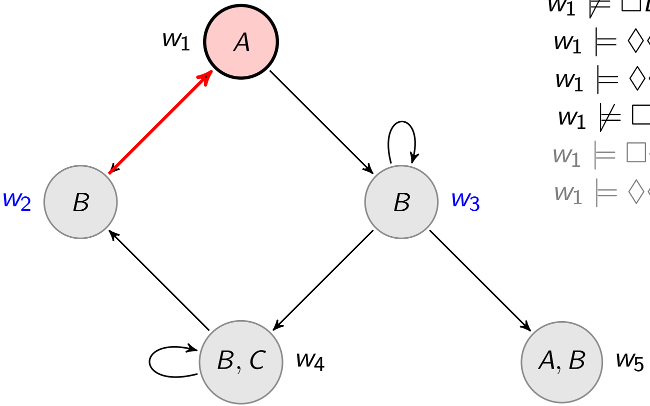

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$w_1 \not\models \Box\Box B$

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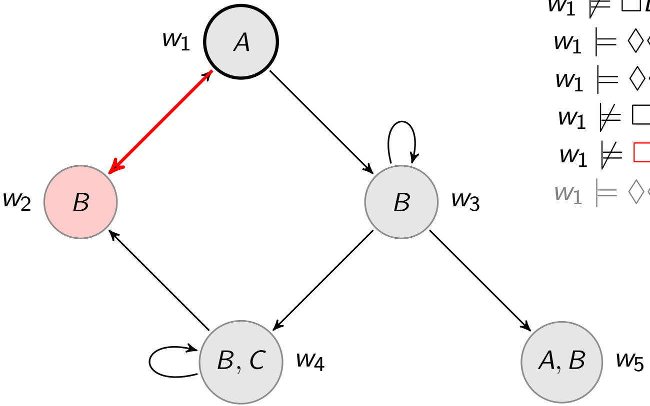

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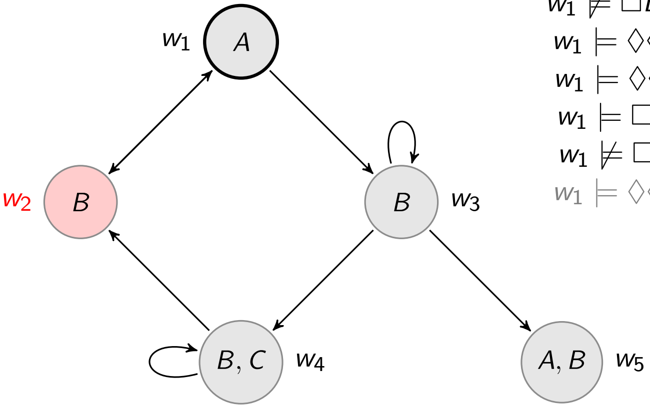

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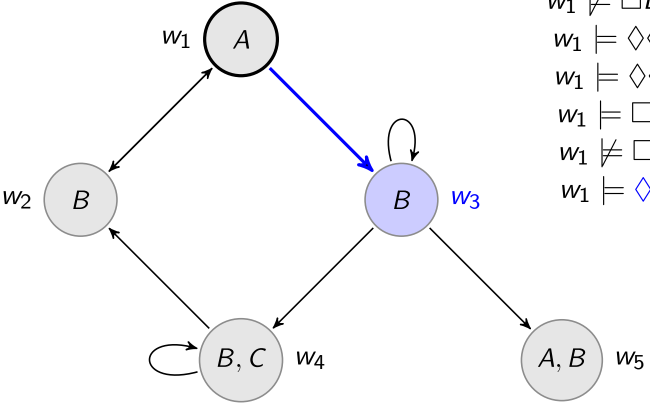
$w_1 \models \Diamond\Diamond C?$



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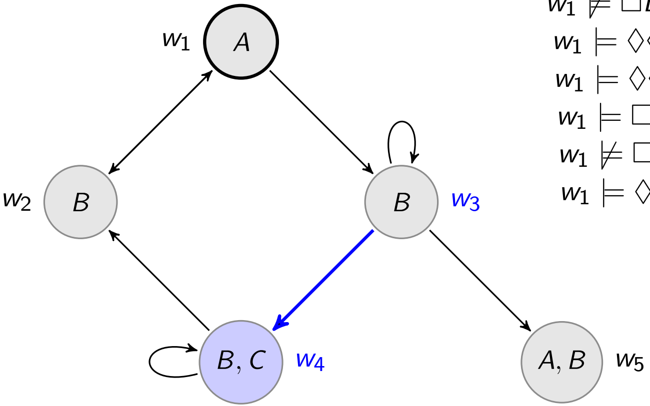

$w_1 \not\models \Box B \wedge B$

$w_1 \models \Diamond\Diamond B$

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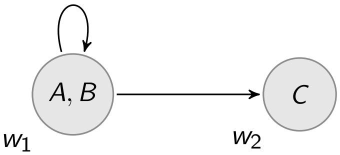
$w_1 \models \Box\Box B$

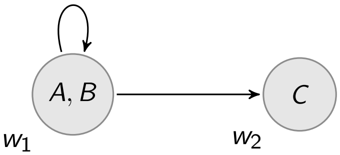
$w_1 \not\models \Box\Diamond C$

$w_1 \models \Diamond\Diamond C$

$\Box(A \to B)$ vs. $A \to \Box B$

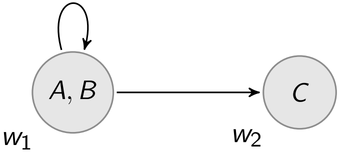


$\Box(A \to B)$ vs. $A \to \Box B$



$w \models X \to Y$ provided either $w \not\models X$ or $w \models Y$

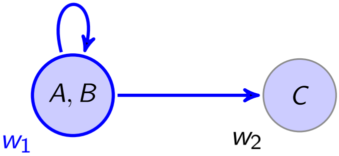
$\Box(A \to B)$ vs. $A \to \Box B$



$w_1 \models \Box(A \to B)$

$w \models X \to Y$ provided either $w \not\models X$ or $w \models Y$

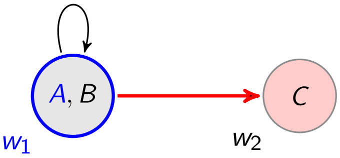
$\Box(A \to B)$ vs. $A \to \Box B$



$w_1 \models \Box(A \to B)$

$w \models X \to Y$ provided either $w \not\models X$ or $w \models Y$

$\Box(A \to B)$ vs. $A \to \Box B$



$w_1 \models \Box(A \to B)$ and $w_1 \not\models A \to \Box B$

$w \models X \to Y$ provided either $w \not\models X$ or $w \models Y$

Some Facts

- $\Box\varphi \vee \neg\Box\varphi$ is always true (i.e., true at any state in any Kripke structure), but what about $\Box\varphi \vee \Box\neg\varphi$?

Some Facts

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- $\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$ is true at any state in any Kripke structure.

More Facts

Determine which of the following formulas are *always* true at any state in any Kripke structure:

1. $\Box\varphi \to \Diamond\varphi$
2. $\Box(\varphi \lor \neg\varphi)$
3. $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$
4. $\Box\varphi \to \varphi$
5. $P \to \Box\Diamond\varphi$
6. $\Diamond(\varphi \lor \psi) \to \Diamond\varphi \lor \Diamond\psi$

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Eg., for each state w, w is accessible from itself (R is a reflexive relation).

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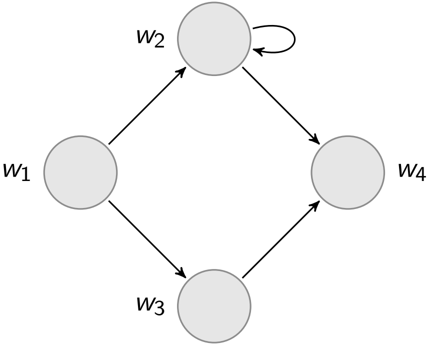
Some Facts

- $\Box\varphi \rightarrow \varphi$ is true at any state in any Kripke structure where each state is accessible from itself.
- $\Box\varphi \rightarrow \Diamond\varphi$ is true at any state in any Kripke structure where each state has at least one accessible world.

Can you think of properties that force each of the following formulas to be true at any state in any appropriate Kripke structure?

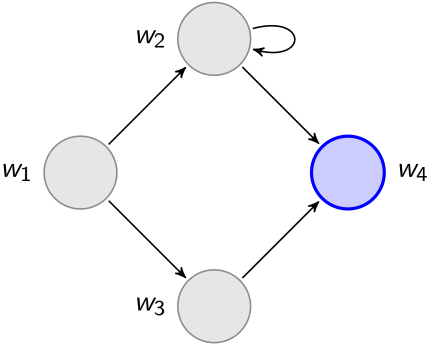
1. $\Diamond\varphi \to \Box\varphi$
2. $\Box\varphi \to \Box\Box\varphi$

Defining States



- $w_4 \models$
- $w_3 \models$
- $w_2 \models$
- $w_1 \models$

Defining States



- $w_4 \models \Box\bot$
- $w_3 \models$
- $w_2 \models$
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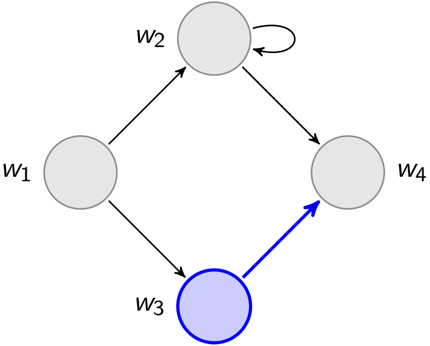

- $w_4 \models \Box\bot$
- $w_3 \models \Diamond\Box\bot \wedge \Box\Box\bot$
- $w_2 \models$
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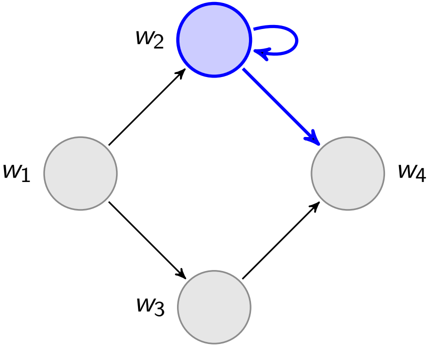
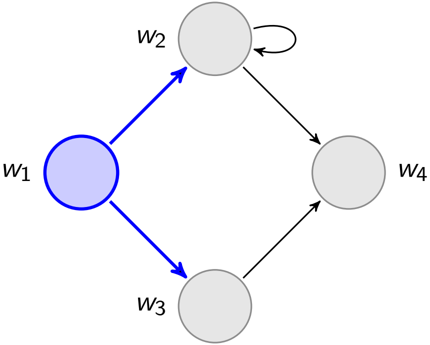
Defining States



- $w_4 \models \Box\bot$
- $w_3 \models \Diamond\Box\bot \land \Box\Box\bot$
- $w_2 \models \Diamond\Box\bot \land \Diamond\Diamond\top$
- $w_1 \models$

Defining States



- $w_4 \models \Box\bot$
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- $w_1 \models \Diamond(\Diamond\Box\bot \wedge \Box\Box\bot)$

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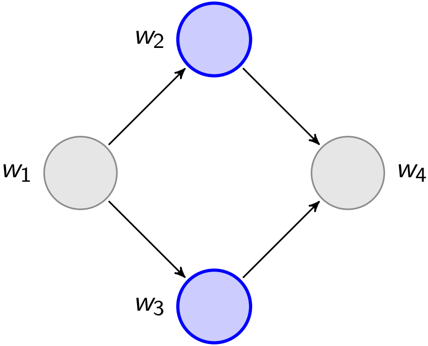

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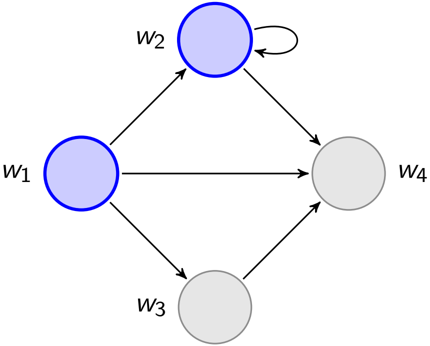

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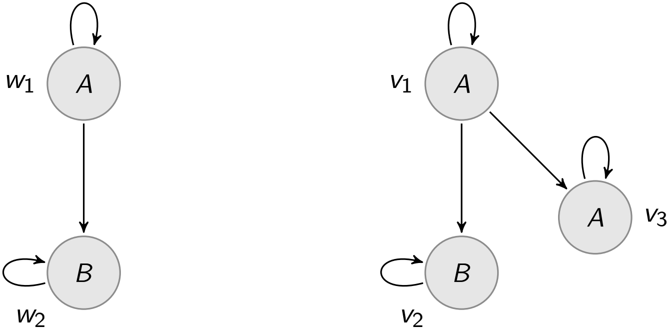
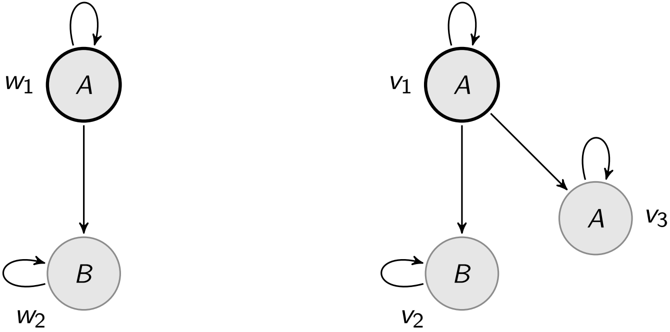
Distinguishing States



What is the difference between states w_1 and v_1?

Distinguishing States



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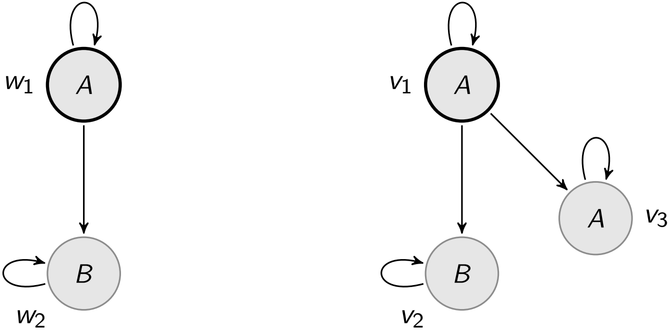
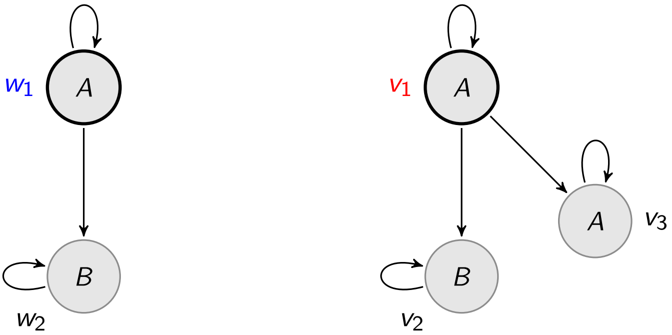


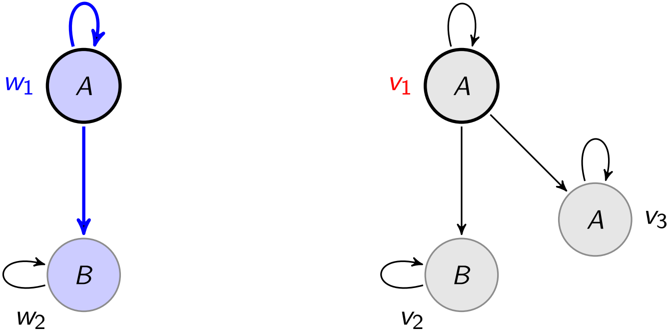
Is there a modal formula true at w_1 but not at v_1?

Distinguishing States



$w_1 \models \Box\Diamond\neg A$ but $v_1 \not\models \Box\Diamond\neg A$.

Distinguishing States



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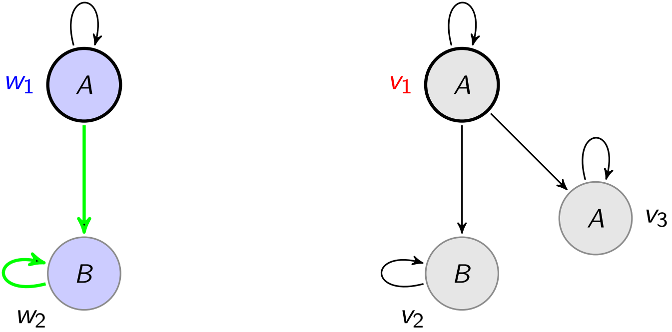

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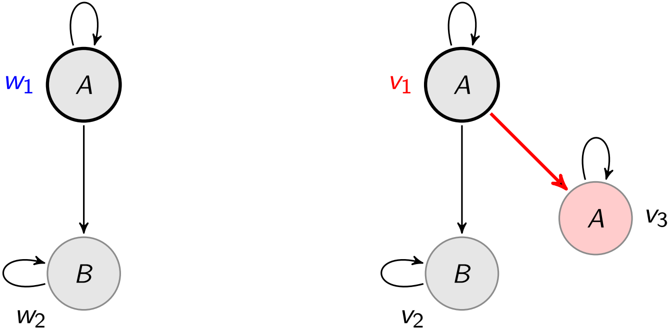

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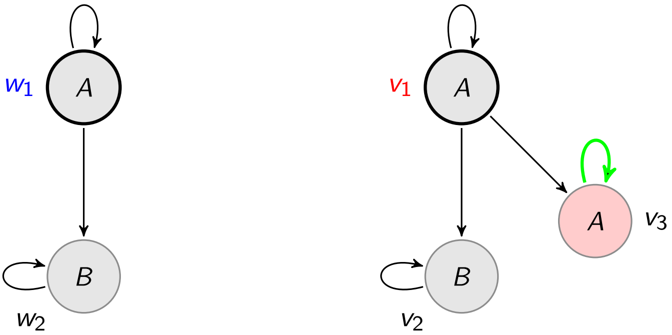

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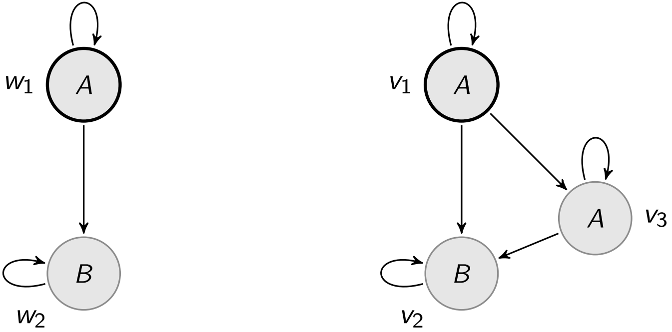
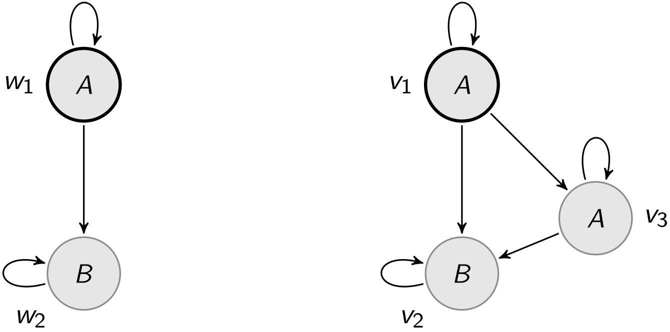


What about now? Is there a modal formula true at w_1 but not v_1?

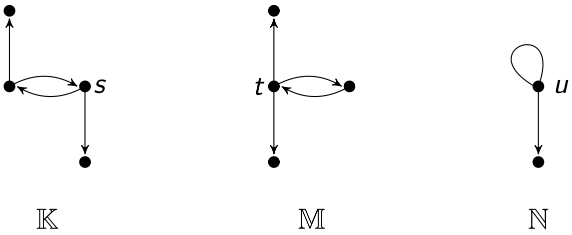
Distinguishing States



No modal formula can distinguish w_1 and v_1!

A More Complicated Example

Which pair of states cannot be distinguished by a modal formula?



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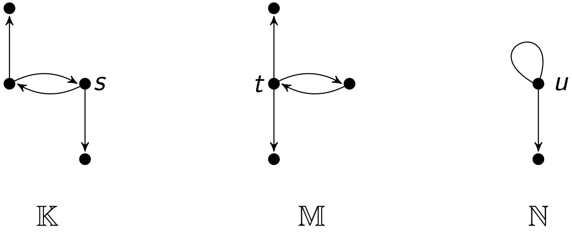

$$\Box(\Box\bot \vee \Diamond\Box\bot)$$

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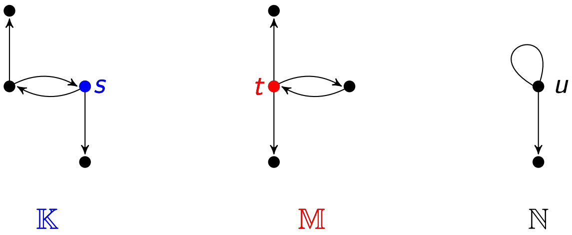

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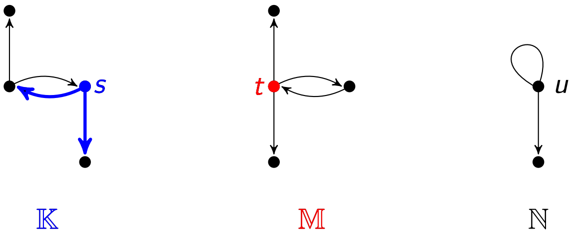

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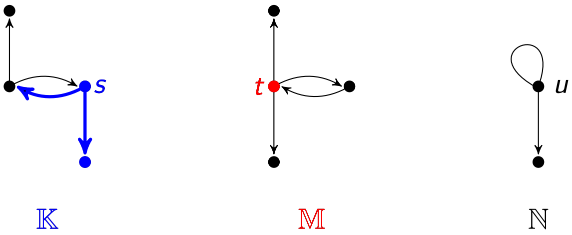

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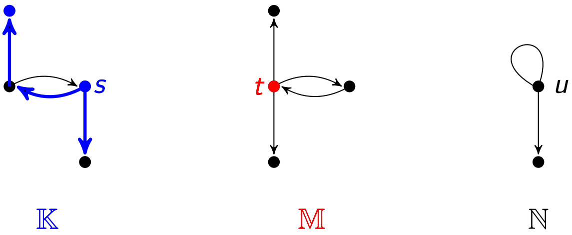

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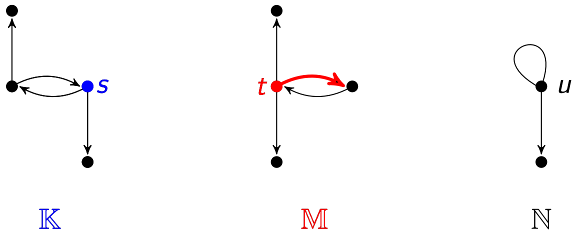

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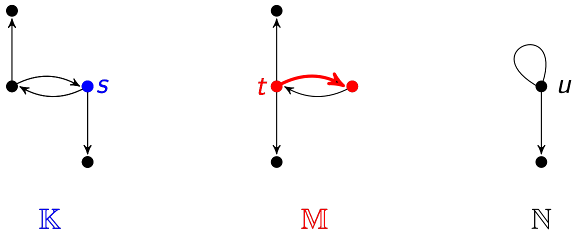

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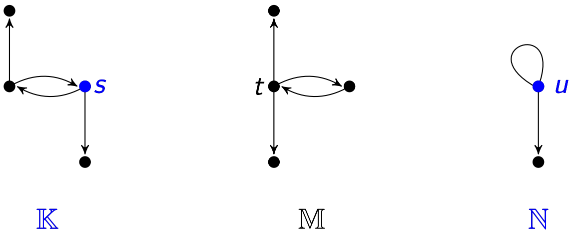

Next time: Chapters 3 & 4.

Questions?

Email: epacuit@umd.edu
Website: ai.stanford.edu/~epacuit
Office: Skinner 1103A