An Incomplete Modal Logic
Notes for Lecture 8

Eric Pacuit*

April 3, 2012

Fact 1 *$\mathcal{F} \models (\Diamond\varphi \wedge \Diamond\psi) \to (\Diamond(\varphi \wedge \Diamond\psi) \vee \Diamond(\varphi \wedge \psi) \vee \Diamond(\Diamond\varphi \wedge \psi))$ iff $\mathcal{F}$ non-branching to the right (for all w, v, x if wRv and wRx then either vRx or xRv or $v = x$).*

Fact 2 *$\mathcal{F} \models \Box\varphi \to \Diamond\varphi$ iff $\mathcal{F}$ is unbounded (to the right: for all w there is a v such that wRv).*

Fact 3 *$\mathcal{F} \models \Box(\Box\varphi \to \varphi) \to \Box\varphi$ iff $\mathcal{F}$ is transitive and converse well-founded.*

Temporal Logic: Let $\mathcal{M} = \langle T, R, V\rangle$ be a Kripke model.

- $\mathcal{M}, t \models F\varphi$ iff there exists a t' such that tRt' and $\mathcal{M}, t' \models \varphi$
- $\mathcal{M}, t \models P\varphi$ iff there exists a t' such that $t'Rt$ and $\mathcal{M}, t' \models \varphi$
- $\mathcal{M}, t \models G\varphi$ iff for all t', if tRt' then $\mathcal{M}, t' \models \varphi$
- $\mathcal{M}, t \models H\varphi$ iff for all t', if $t'Rt$ then $\mathcal{M}, t' \models \varphi$

The minimal temporal logic $\mathbf{K_t}$ contains the following axiom schemes and rules:

- Propositional logic
- $G(\varphi \to \psi) \to (G\varphi \to G\psi)$
- $G(\varphi \to \psi) \to (G\varphi \to G\psi)$
- $\varphi \to GP\varphi$
- $\varphi \to HF\varphi$
- From φ derive $\bigcirc\varphi$ where $\bigcirc = G, H$
- Modus Ponens

* *Webpage*: ai.stanford.edu/∼epacuit, *Email*: e.j.pacuit@uvt.nl

Let $\mathbf{K_t Tho}$ be the temporal logic extending $\mathbf{K_t}$ with the axiom schemes

- $Fp \wedge Fq \rightarrow (F(p \wedge Fq) \vee F(p \wedge q) \vee F(Fp \wedge q))$
- $Gp \rightarrow Fp$
- $H(Hp \rightarrow p) \rightarrow Hp$

Fact 4 $\mathbf{K_t Tho}$ *is consistent.*

Fact 5 *If* $\mathcal{F} = \langle T, R \rangle$ *is a frame for* $\mathbf{K_t Tho}$*, then for* $t \in T$*,* $\{u \mid tRu\}$ *is an unbounded strict total order.*

Fact 6 *If* $\mathcal{F} = \langle T, R \rangle$ *is a frame for* $\mathbf{K_t Tho}$*, then* $\mathcal{F} \not\models GFp \rightarrow FGp$*.*

Fact 7 *The logic* $\mathbf{K_t ThoM}$ *which extends* $\mathbf{K_t Tho}$ *with the axiom scheme* $GF\varphi \rightarrow FG\varphi$ *is* consistent *and* incomplete *(I.e.,* $\mathbf{K_t ThoM}$ *is not the logic for any class of frames).*

Why is $\mathbf{K_t ThoM}$ *consistent?*

General Frames: A general frame is a tuple $\langle W, R, A \rangle$ where A is a collection of *admissible* subsets of W closed under the following operations:

- If $X, Y \in A$ then $X \cap Y \in A$
- If $X \in A$ then $\overline{X} \in A$
- If $X \in A$ then $l(X) = \{w \mid \text{for all } v \text{ if } wRv \text{ then } v \in X\} \in A$.

A model based on a general frame $\mathcal{F} = \langle W, R, A \rangle$ is a tuple $\mathcal{M} = \langle W, R, A, V \rangle$ where for each $p \in \mathsf{At}$, $V(p) \in A$.

Fact 8 *The McKinsey axiom* $\Box\Diamond p \rightarrow \Diamond\Box p$ *is not valid on* $\langle \mathbb{N}, < \rangle$*, but it is valid on* $\mathbb{N}, <, A \rangle$*, where* A *contains all the finite and co-finite subsets of* $\mathbb{N}$*.*

Fact 9 $\langle \mathbb{N}, <, A \rangle$*, where* A *contains all finite and co-finite sets, is a general frame satisfying all the axiom schemes and rules in* $\mathbf{K_t ThoM}$*.*