Basic Correspondence Theory
Notes for Lecture 7

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Definition 1 (Frame) A pair $\langle W, R\rangle$ with W a nonempty set of states and $R \subseteq W \times W$ is called a **frame**. Given a frame $\mathcal{F} = \langle W, R\rangle$, we say the model $\mathcal{M}$ **is based on the frame** $\mathcal{F} = \langle W, R\rangle$ if $\mathcal{M} = \langle W, R, V\rangle$ for some valuation function V. ◁

Definition 2 (Frame Validity) Given a frame $\mathcal{F} = \langle W, R\rangle$, a modal formula φ is **valid on** $\mathcal{F}$, denoted $\mathcal{F} \models \varphi$, provided $\mathcal{M} \models \varphi$ for all models $\mathcal{M}$ based on $\mathcal{F}$. ◁

Suppose that P is a property of relations (eg., reflexivity or transitivity). We say a frame $\mathcal{F} = \langle W, R\rangle$ has property P provided R has property P. For example,

- $\mathcal{F} = \langle W, R\rangle$ is called a **reflexive frame** provided R is reflexive, i.e., for all $w \in W$, wRw.
- $\mathcal{F} = \langle W, R\rangle$ is called a **transitive frame** provided R is transitive, i.e., for all $w, x, v \in W$, if wRx and xRv then wRv.

Definition 3 (Defining a Class of Frames) A modal formula φ **defines the class of frames with property** P provided for all frames $\mathcal{F}$, $\mathcal{F} \models \varphi$ iff $\mathcal{F}$ has property P. ◁

Remark 4 *Note that if $\mathcal{F} \models \varphi$ where φ is some modal formula, then $\mathcal{F} \models \varphi^*$ where φ^* is any* **substitution instance** *of φ. That is, φ^* is obtained by replacing sentence letters in φ with modal formulas. In particular, this means, for example, that in order to show that $\mathcal{F} \not\models \Box\varphi \rightarrow \varphi$ it is enough to show that $\mathcal{F} \not\models \Box p \rightarrow p$ where p is a sentence letter. (This will be used in the proofs below).*

Fact 5 *$\Box\varphi \rightarrow \varphi$ defines the class of reflexive frames.*

Proof. We must show for any frame $\mathcal{F}$, $\mathcal{F} \models \Box\varphi \rightarrow \varphi$ iff $\mathcal{F}$ is reflexive.

($\Leftarrow$) Suppose that $\mathcal{F} = \langle W, R\rangle$ is reflexive and let $\mathcal{M} = \langle W, R, V\rangle$ be any model based on $\mathcal{F}$. Given $w \in W$, we must show $\mathcal{M}, w \models \Box\varphi \rightarrow \varphi$. Suppose that $\mathcal{M}, w \models \Box\varphi$. Then for all $v \in W$, if wRv then $\mathcal{M}, v \models \varphi$. Since R is reflexive, we have wRw. Hence, $\mathcal{M}, w \models \varphi$. Therefore, $\mathcal{M}, w \models \Box\varphi \rightarrow \varphi$, as desired.

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($\Rightarrow$) We argue by contraposition. Suppose that $\mathcal{F}$ is not reflexive. We must show $\mathcal{F} \not\models \Box\varphi \to \varphi$. By the above Remark, it is enough to show $\mathcal{F} \not\models \Box p \to p$ for some sentence letter p. Since $\mathcal{F}$ is not reflexive, there is a state $w \in W$ such that it is not the case that wRw. Consider the model $\mathcal{M} = \langle W, R, V \rangle$ based on $\mathcal{F}$ with $V(p) = \{v \mid v \neq w\}$. Then $\mathcal{M}, w \models \Box p$ since, by assumption, for all $v \in W$ if wRv, then $v \neq w$ and so $v \in V(p)$. Also, notice that by the definition of V, $\mathcal{M}, w \not\models p$. Therefore, $\mathcal{M}, w \models \Box p \wedge \neg p$, and so, $\mathcal{F} \not\models \Box p \to p$.

($\Rightarrow$, *directly*) Suppose that $\mathcal{F} \models \Box\varphi \to \varphi$. We must show that for all x if xRx. Let x be any state and consider a model $\mathcal{M}$ based on $\mathcal{F}$ with a valuation $V(p) = \{u \mid xRu\}$. Since $\Box p$ is true at x we also have p true at x. This means that $x \in V(p)$, hence, xRx. QED

Fact 6 $\Box\varphi \to \Box\Box\varphi$ *defines the class of transitive frames.*

Proof. We must show for any frame $\mathcal{F}$, $\mathcal{F} \models \Box\varphi \to \Box\Box\varphi$ iff $\mathcal{F}$ is transitive.

($\Leftarrow$) Suppose that $\mathcal{F} = \langle W, R \rangle$ is transitive and let $\mathcal{M} = \langle W, R, V \rangle$ be any model based on $\mathcal{F}$. Given $w \in W$, we must show $\mathcal{M}, w \models \Box\varphi \to \Box\Box\varphi$. Suppose that $\mathcal{M}, w \models \Box\varphi$. We must show $\mathcal{M}, w \models \Box\Box\varphi$. Suppose that $v \in W$ and wRv. We must show $\mathcal{M}, v \models \Box\varphi$. To that end, let $x \in W$ be any state with vRx. Since R is transitive and wRv and vRx, we have wRx. Since $\mathcal{M}, w \models \Box\varphi$, we have $\mathcal{M}, x \models \varphi$. Therefore, since x is an arbitrary state accessible from v, $\mathcal{M}, v \models \Box\varphi$. Hence, $\mathcal{M}, w \models \Box\Box\varphi$, and so, $\mathcal{M}, w \models \Box\varphi \to \Box\Box\varphi$, as desired.

($\Rightarrow$, *by contraposition*) We argue by contraposition. Suppose that $\mathcal{F}$ is not transitive. We must show $\mathcal{F} \not\models \Box\varphi \to \Box\Box\varphi$. By the above Remark, it is enough to show $\mathcal{F} \not\models \Box p \to \Box\Box p$ for some sentence letter p. Since $\mathcal{F}$ is not transitive, there are states $w, v, x \in W$ with wRv and vRx but it is not the case that wRx. Consider the model $\mathcal{M} = \langle W, R, V \rangle$ based on $\mathcal{F}$ with $V(p) = \{y \mid y \neq x\}$. Since $\mathcal{M}, x \not\models p$ and wRv and vRx, we have $\mathcal{M}, w \not\models \Box\Box p$. Furthermore, $\mathcal{M}, w \models \Box p$ since the only state where p is false is x and it is assumed that it is not the case that wRx. Therefore, $\mathcal{M}, w \models \Box p \wedge \neg\Box\Box p$, and so, $\mathcal{F} \not\models \Box p \to \Box\Box p$, as desired.

($\Rightarrow$, *directly*) Suppose that $\mathcal{F} \models \Box\varphi \to \Box\Box\varphi$. We must show that for all x, y, z if xRy and yRz then xRz. Let x be any state and consider a model $\mathcal{M}$ based on $\mathcal{F}$ with a valuation $V(p) = \{u \mid xRu\}$. Since $\Box p$ is true at x we also have $\Box\Box p$ true at x. This means that for all y if xRy then (for all z if yRz we have $z \in V(p)$). Recall that $z \in V(p)$ means that xRz. Putting everything together we have: for all y if xRy then for all z if yRz then xRz. QED

Fact 7 $\varphi \to \Box\Diamond\varphi$ *defines the class of symmetric frames.*

Proof. We must show for any frame $\mathcal{F}$, $\mathcal{F} \models \varphi \to \Box\Diamond\varphi$ iff $\mathcal{F}$ is symmetric.

($\Leftarrow$) Suppose that $\mathcal{F} = \langle W, R \rangle$ is symmetric and let $\mathcal{M} = \langle W, R, V \rangle$ be any model based on $\mathcal{F}$. Given $w \in W$, we must show $\mathcal{M}, w \models \varphi \to \Box\Diamond\varphi$. Suppose that $\mathcal{M}, w \models \varphi$. We must show $\mathcal{M}, w \models \Box\Diamond\varphi$. Suppose that $v \in W$ and wRv. We must show $\mathcal{M}, v \models \Diamond\varphi$. Since R is symmetric and wRv, we have vRw. Since $\mathcal{M}, w \models \varphi$, we have $\mathcal{M}, v \models \Diamond\varphi$. Hence, $\mathcal{M}, w \models \Box\Diamond\varphi$, as desired.

($\Rightarrow$, *by contraposition*) We argue by contraposition. Suppose that $\mathcal{F}$ is not symmetric. We must show $\mathcal{F} \not\models \varphi \to \Box\Diamond\varphi$. By the above Remark, it is enough to show $\mathcal{F} \not\models p \to \Box\Diamond p$ for some sentence letter p. Since $\mathcal{F}$ is not symmetric, there are states $w, v \in W$ with wRv but it is not the case that vRw. Consider the model $\mathcal{M} = \langle W, R, V \rangle$ based on $\mathcal{F}$ with $V(p) = \{w\}$. Then, $\mathcal{M}, w \models p$. Since it is not the case that vRw and w is the only state satisfying p, we have $\mathcal{M}, v \not\models \Diamond p$. This means that $\mathcal{M}, w \not\models \Box\Diamond p$ (since wRv and $\mathcal{M}, v \not\models \Diamond p$).

($\Rightarrow$, *directly*) Suppose that $\mathcal{F} \models \varphi \to \Box\Diamond\varphi$. We must show that for all x, y if xRy then yRx. Let x be any state and consider a model $\mathcal{M}$ based on $\mathcal{F}$ with a valuation $V(p) = \{u \mid u = x\}$. Since p is true at x we also have $\Box\Diamond p$ true at x. This means that for all y if xRy then there is a z such that yRz and $z \in V(p)$. Recall that $z \in V(p)$ means that $z = x$. Putting everything together we have: for all y if xRy then there is a z such that z if yRz then $x = z$. This property is symmetry. QED

Fact 8 *$\Diamond\Box\varphi \to \Box\Diamond\varphi$ defines the confluence property: for all x, y, z if xRy and xRz then there is a s such that yRs and zRs.*

Proof. We must show for any frame $\mathcal{F}$, $\mathcal{F} \models \Diamond\Box\varphi \to \Box\Diamond\varphi$ iff $\mathcal{F}$ satisfies the confluence property: for all x, y, z if xRy and xRz then there is a s such that yRs and zRs.

($\Leftarrow$) Suppose that $\mathcal{F} = \langle W, R \rangle$ satisfies confluence and let $\mathcal{M} = \langle W, R, V \rangle$ be any model based on $\mathcal{F}$. Given $w \in W$, we must show $\mathcal{M}, w \models \Diamond\Box\varphi \to \Box\Diamond\varphi$. Suppose that $\mathcal{M}, w \models \Diamond\Box\varphi$. We must show $\mathcal{M}, w \models \Box\Diamond\varphi$. Suppose that $x \in W$ with wRx. Since $\mathcal{M}, w \models \Diamond\Box\varphi$, there is a y such that wRy and $\mathcal{M}, y \models \Box\varphi$. Since wRx and wRy, by the confluence property, there is a $s \in W$ with xRs and yRs. Since yRs and $\mathcal{M}, y \models \Diamond\varphi$, we have $\mathcal{M}, s \models \varphi$. Then, since xRs, we have $\mathcal{M}, x \models \Diamond\varphi$. Hence, $\mathcal{M}, w \models \Box\Diamond\varphi$, as desired.

($\Rightarrow$, *by contraposition*) We argue by contraposition. Suppose that $\mathcal{F}$ does not satisfy confluence. We must show $\mathcal{F} \not\models \Diamond\Box\varphi \to \Box\Diamond\varphi$. By the above Remark, it is enough to show $\mathcal{F} \not\models \Diamond\Box p \to \Box\Diamond p$ for some sentence letter p. Since $\mathcal{F}$ does not satisfy confluence, there are states $w, x, y \in W$ with wRx and wRy but there is no s such that xRs and yRs. Consider the model $\mathcal{M} = \langle W, R, V \rangle$ based on $\mathcal{F}$ with $V(p) = \{v \mid yRv\}$. Then, $\mathcal{M}, y \models \Box p$ (since all states accessible from y satisfy p). Since there is no s such that xRs and yRs, we also have $\mathcal{M}, x \not\models \Diamond p$. Since wRx and wRy, we have $\mathcal{M}, w \not\models \Box\Diamond p$ and $\mathcal{M}, w \models \Diamond\Box p$. Hence, $\Diamond\Box p \to \Box\Diamond p$ is not valid.

($\Rightarrow$, *directly*) Suppose that $\mathcal{F} \models \Diamond\Box\varphi \to \Box\Diamond\varphi$. We must show that for all x, y, z if xRy and xRz, then there is a s such that yRs and zRs. Let x be any state and consider a model $\mathcal{M}$ based on $\mathcal{F}$ with a valuation $V(p) = \{u \mid yRu\}$. Let y, z be states with xRy and xRz. Since, $\mathcal{M}, y \models \Box p$, we have $\mathcal{M}, x \models \Diamond\Box p$. This means that $\mathcal{M}, x \models \Box\Diamond p$. Hence, since xRz, we have $\mathcal{M}, z \models \Diamond p$. Thus, there is a states v such that zRv and $v \in V(p)$. Since $v \in V(p)$, we have yRv. Putting everything together we have: for all x, y, z if xRy and xRz, then there is a s such that yRs and zRs. QED

Theorem 9 (Goldblatt-Thomason) *A first-order definable class K of frames is modally definable iff it is closed under taking bounded morphic images, generated subframes, disjoint unions and reflects ultrafilter extensions.*

Sahlqvist's Algorithm (see section 9.3 of *Modal Logic for Open Minds* and Sections 3.5 - 3.7 of *Modal Logic* by Blackburn, de Rijke and Venema for an extensive discussion).

Not all modal formulas correspond to first-order properties:

Basic properties of first-order logic:

- **Compactness**: Γ is satisfiable iff every finite subset is satisfiable.
- **Löwenheim-Skolem Theorem**: If Γ is satisfiable, then it is satisfiable on a countable model.

Fact 10 $\mathcal{F} \models \Box(\Box\varphi \to \varphi) \to \Box\varphi$ *iff* $\mathcal{F}$ *is transitive and converse well-founded.*

Fact 11 $\Box\Diamond\varphi \to \Diamond\Box\varphi$ *does not correspond to a first-order condition.*