Filtrations and Basic Proof Theory
Notes for Lecture 5

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1 Filtration

Let $\mathcal{M} = \langle W, R, V \rangle$ be a Kripke model. Suppose that Σ is a set of formulas closed under subformulas. We write say w and v are Σ-equivalent provided:

$$w \leftrightsquigarrow_\Sigma v \text{ iff for all } \varphi \in \Sigma,\ \mathcal{M}, w \models \varphi \text{ iff } \mathcal{M}, v \models \varphi.$$

Note that $\leftrightsquigarrow_\Sigma$ is an equivalence relation. Let $|w|_\Sigma = \{v \mid w \leftrightsquigarrow_\Sigma v\}$ denote the equivalence class of w under $\leftrightsquigarrow_\Sigma$.

Definition 1.1 (Filtration) Let $\mathcal{M} = \langle W, R, V \rangle$ be a Kripke model. Given a set of formulas closed under subformulas, a model $\mathcal{M}^f = \langle W^f, R^f, V^f \rangle$ is said to be a filtration of $\mathcal{M}$ through Σ provided

- $W^f = \{|w|_\Sigma \mid w \in W\}$
- If wRv then $|w|_\Sigma R^f |v|_\Sigma$
- If $|w|_\Sigma R^f |v|_\Sigma$ then for each $\Diamond\varphi \in \Sigma$, if $\mathcal{M}, v \models \varphi$ then $\mathcal{M}, w \models \Diamond\varphi$
- $V^f(p) = \{|w|_\Sigma \mid w \in V(p)\}$ $\triangleleft$

Theorem 1.2 *If $\mathcal{M}^f$ is a filtration of $\mathcal{M}$ through Σ, then for all $\varphi \in \Sigma$,*

$$\mathcal{M}, w \models \varphi \quad \textit{iff} \quad \mathcal{M}^f, |w|_\Sigma \models \varphi$$

Examples of Filtrations

- **smallest filtration**: $|w|_\Sigma R^s |v|_\Sigma$ iff there is $w' \in |w|_\Sigma$ and $v' \in |v|_\Sigma$ such that $w'Rv'$.
- **largest filtration**: $|w|_\Sigma R^l |v|_\Sigma$ iff for all $\Diamond\varphi \in \Sigma$, $\mathcal{M}, v \models \varphi$ implies $\mathcal{M}, w \models \Diamond\varphi$
- **transitive filtration**: $|w|_\Sigma R^t |v|_\Sigma$ iff for all $\Diamond\varphi \in \Sigma$, $\mathcal{M}, v \models \varphi \vee \Diamond\varphi$ implies $\mathcal{M}, w \models \Diamond\varphi$ (assuming R is transitive)

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2 The Minimal Modal Logic

For a complete discussion of this material, consult Chapter 5 of *Modal Logic for Open Minds* by Johan van Benthem.

Definition 2.1 (Substitution) A **substitution** is a function from sentence letters to well formed modal formulas (i.e., $\sigma : \mathsf{At} \to WFF_{ML}$). We extend a substitution σ to all formulas φ by recursion as follows (we write φ^σ for $\sigma(\varphi)$):

1. $\sigma(\bot) = \bot$
2. $\sigma(\neg\varphi) = \neg\sigma(\varphi)$
3. $\sigma(\varphi \wedge \psi) = \sigma(\varphi) \wedge \sigma(\psi)$
4. $\sigma(\Box\varphi) = \Box\sigma(\varphi)$
5. $\sigma(\Diamond\varphi) = \Diamond\sigma(\varphi)$ ◁

For example, if $\sigma(p) = \Box\Diamond(p \wedge q)$ and $\sigma(q) = p \wedge \Box q$ then

$$(\Box(p \wedge q) \to \Box p)^\sigma = \Box((\Box\Diamond(p \wedge q)) \wedge (p \wedge \Box q)) \to \Box(\Box\Diamond(p \wedge q))$$

Definition 2.2 (Tautology) A modal formula φ is called a **(propositional) tautology** if $\varphi = (\alpha)^\sigma$ where σ is a substition, α is a formula of propositional logic and α is a tautology. ◁

For example, $\Box p \to (\Diamond(p \wedge q) \to \Box p)$ is a tautology because $a \to (b \to a)$ is a tautology in the language of propositional logic and

$$(a \to (b \to a))^\sigma = \Box p \to (\Diamond(p \wedge q) \to \Box p)$$

where $\sigma(a) = \Box p$ and $\sigma(b) = \Diamond(p \wedge q)$.

Definition 2.3 (Modal Deduction) A **modal deduction** is a finite sequence of formulas $\langle \alpha_1, \ldots, \alpha_n \rangle$ where for each $i \leq n$ either

1. α_i is a tautology
2. α_i is a substitution instance of $\Box(p \to q) \to (\Box p \to \Box q)$
3. α_i is of the form $\Box\alpha_j$ for some $j < i$
4. α_i follows by modus ponens from earlier formulas (i.e., there is $j, k < i$ such that α_k is of the form $\alpha_j \to \alpha_i$).

We write $\vdash_{\mathbf{K}} \varphi$ if there is a deduction containing φ. ◁

The formula in item 2. above is called the **K axiom** and the application of item 3. is called the rule of **necessitation**.

Fact 2.4 $\vdash_{\mathbf{K}} \Box(\varphi \wedge \psi) \to (\Box\varphi \wedge \Box\psi)$

Proof.

1.	$\varphi \wedge \psi \to \varphi$	tautology
2.	$\Box((\varphi \wedge \psi) \to \varphi)$	Necessitation 1
3.	$\Box((\varphi \wedge \psi) \to \varphi) \to (\Box(\varphi \wedge \psi) \to \Box\varphi)$	Substitution instance of K
4.	$\Box(\varphi \wedge \psi) \to \Box\varphi$	MP 2,3
5.	$\varphi \wedge \psi \to \psi$	tautology
6.	$\Box((\varphi \wedge \psi) \to \psi)$	Necessitation 5
7.	$\Box((\varphi \wedge \psi) \to \varphi) \to (\Box(\varphi \wedge \psi) \to \Box\psi)$	Substitution instance of K
8.	$\Box(\varphi \wedge \psi) \to \Box\psi$	MP 5,6
9.	$(a \to b) \to ((a \to c) \to (a \to (b \wedge c)))$	tautology $(a := \Box(\varphi \wedge \psi), b := \Box\varphi, c := \Box\psi)$
10.	$(a \to c) \to (a \to (b \wedge c))$	MP 4,9
11.	$\Box(\varphi \wedge \psi) \to \Box\varphi \wedge \Box\psi$	MP 8,10

QED

Fact 2.5 *If* $\vdash_{\mathbf{K}} \varphi \to \psi$ *then* $\vdash_{\mathbf{K}} \Box\varphi \to \Box\psi$

Proof.

1.	$\varphi \to \psi$	assumption
2.	$\Box(\varphi \to \psi)$	Necessitation 1
3.	$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$	Substitution instance of K
4.	$\Box\varphi \to \Box\psi$	MP 2,3

QED

Definition 2.6 (Modal Deduction with Assumptions) Let Σ be a set of modal formulas. A **modal deduction of** φ **from** Σ, denoted $\Sigma \vdash_{\mathbf{K}} \varphi$ is a finite sequence of formulas $\langle \alpha_1, \ldots, \alpha_n \rangle$ where for each $i \leq n$ either

1. α_i is a tautology

2. $\alpha_i \in \Sigma$

3. α_i is a substitution instance of $\Box(p \to q) \to (\Box p \to \Box q)$

4. α_i is of the form $\Box\alpha_j$ for some $j < i$ and $\vdash_{\mathbf{K}} \alpha_j$

5. α_i follows by modus ponens from earlier formulas (i.e., there is $j, k < i$ such that α_k is of the form $\alpha_j \to \alpha_i$). ◁

Remark 2.7 *Note that the side condition in item 4. in the above definition is crucial. Without it, one application of Necessitation shows that* $\{p\} \vdash_{\mathbf{K}} \Box p$. *Using the general fact (cf. Exercise #4, Section 1.2 of Enderton) that* $\Sigma; \alpha \vdash_{\mathbf{K}} \beta$ *implie* $\Sigma \vdash_{\mathbf{K}} \alpha \to \beta$, *we can conclude that* $\vdash_{\mathbf{K}} p \to \Box p$. *But, clearly* $p \to \Box p$ *cannot be a theorem (why?).*

Definition 2.8 (Logical Consequence) Suppose that Σ is a set of modal formulas. We say φ is a **logical consequence** of Σ, denoted $\Sigma \models \varphi$ provided for all frames $\mathcal{F}$, if $\mathcal{F} \models \alpha$ for each $\alpha \in \Sigma$, then $\mathcal{F} \models \varphi$. ◁

Theorem 2.9 (Soundness) *If* $\Sigma \vdash_{\mathbf{K}} \varphi$ *then* $\Sigma \models \varphi$.

Proof. The proof is by induction on the length of derivations. See Chapter 5 in *Modal Logic for Open Minds* and your lecture notes. QED

Theorem 2.10 (Completeness) *If* $\Sigma \models \varphi$ *then* $\Sigma \vdash_{\mathbf{K}} \varphi$.

Proof. See Chapter 5 in *Modal Logic for Open Minds* and your lecture notes for a proof. QED

Remark 2.11 (Alternative Statement of Soundness and Completeness) *Suppose that* Σ *is a set of modal formulas. Define the minimal modal logic as the smallest set* $\Lambda_{\mathbf{K}}(\Sigma)$ *of modal formulas extending* Σ *that (1) contains all tautologies, (2) contains the formula* $\Box(p \to q) \to (\Box p \to \Box q)$*, (3) is closed under substitutions, (4) is closed under the Necessitation rule (i.e., if* $\varphi \in \Lambda_{\mathbf{K}}$ *is derivable* without premises – $\vdash_{\mathbf{K}} \varphi$ – *then* $\Box\varphi \in \Lambda_{\mathbf{K}}$*) and (4) is closed under Modus Ponens. Suppose* $\mathfrak{F}(\Sigma) = \{\varphi \mid \Sigma \models \varphi\}$*. Then, soundness and completeness states that* $\Lambda_{\mathbf{K}}(\Sigma) = \mathfrak{F}(\Sigma)$.

Some Axioms		**Some Modal Logics**	
K	$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$	**K**	$K + PC + Nec$
D	$\Box\varphi \to \Diamond\varphi$	**T**	$K + T + PC + Nec$
T	$\Box\varphi \to \varphi$	**S4**	$K + T + 4 + PC + Nec$
4	$\Box\varphi \to \Box\Box\varphi$	**S5**	$K + T + 4 + 5 + PC + Nec$
5	$\neg\Box\varphi \to \Box\neg\Box\varphi$	**KD45**	$K + D + 4 + 5 + PC + Nec$
W	$\Box(\Box\varphi \to \varphi) \to \Box\varphi$	**GL**	$K + W$

Completeness Theorems

- **T** is sound and strongly complete with respect to the class reflexive Kripke frames.
- **S4** is sound and strongly complete with respect to the class reflexive Kripke frames.
- **S5** is sound and strongly complete with respect to the class reflexive Kripke frames.
- **KD45** is sound and strongly complete with respect to the class reflexive Kripke frames.

3 Alternative Proof of Weak Completeness

In this section we illustrate a technique for by proving weak completeness invented by Larry Moss in [1]. Since we are only interested in illustrating the technique, we focus on the smallest normal modal logic (**K**). Recall that the basic modal language is generated by the following grammar:

$$p \mid \neg\varphi \mid \varphi \wedge \psi \mid \Diamond\varphi$$

where p is a propositional variable (let $\mathsf{At} = \{p_1, p_2, \dots, p_n, \dots\}$ deonte the set of propositional variables). Define the usual boolean connectives and the modal operator $\Box$ as usual. Let $\mathcal{L}_\Diamond$ be the set of well-formed formulas.

Some notation is useful at this stage. The **height**, or **modal depth**, of a formula $\varphi \in \mathcal{L}_\Diamond$, denoted $\mathsf{ht}(\varphi)$, is longest sequence of nested modal operators. Formally, define ht as follows

$$\begin{array}{lcl} \mathsf{ht}(p_n) & = & 0 \\ \mathsf{ht}(\neg\varphi) & = & \mathsf{ht}(\varphi) \\ \mathsf{ht}(\varphi \vee \psi) & = & \max\{\mathsf{ht}(\varphi), \mathsf{ht}(\psi)\} \\ \mathsf{ht}(\Diamond\varphi) & = & 1 + \mathsf{ht}(\varphi) \end{array}$$

The **order** of a modal formula φ, written $\mathsf{ord}(\varphi)$, is the largest index of a propositional formula that appears in φ. Formally,

$$\begin{array}{lcl} \mathsf{ord}(p_n) & = & n \\ \mathsf{ord}(\neg\varphi) & = & \mathsf{ord}(\varphi) \\ \mathsf{ord}(\varphi \vee \psi) & = & \max\{\mathsf{ord}(\varphi), \mathsf{ord}(\psi)\} \\ \mathsf{ord}(\Diamond_n\varphi) & = & \mathsf{ord}(\varphi) \end{array}$$

Let $\mathcal{L}_{h,n} = \{\varphi \mid \varphi \in \mathcal{L}_\Diamond,\ \mathsf{ht}(\varphi) \leq h \text{ and } \mathsf{ord}(\varphi) \leq n\}$. Thus, for example, $\mathcal{L}_{0,n}$ is the propositional language (finite up to logical equivalence) built from the set $\{p_1, \ldots, p_n\}$ of propositional variables.

A set $T \subseteq \{p_1, \ldots, p_m\}$ corresponds to a partial valuation on At if we think of the elements of T as being true and the elements of $\{p_1, \ldots, p_m\} - T$ as being false. This partial valuation can be described by the following formula of $\mathcal{L}_{0,m}$

$$\widehat{T} = \bigwedge_{p \in T} p \wedge \bigwedge_{p \in \{p_1, \ldots, p_n\} - T} \neg p$$

Now, for each $\varphi \in \mathcal{L}_{0,m}$ it is easy to see that exactly one of the following holds: $\vdash \widehat{T} \to \varphi$ or $\vdash \widehat{T} \to \neg\varphi$. Furthermore, it is easy to show that for each $\varphi \in \mathcal{L}_{0,m}$, $\vdash \varphi \leftrightarrow \bigvee\{\widehat{T} \mid \vdash \widehat{T} \to \varphi\}$. The central idea of Moss' technique is to generalize these facts to modal logic.

It is well-known that modal logic has the *finite tree property*, i.e., when evaluating a formula φ it is enough to consider only paths of length at most the modal depth of φ. The modal generalization of the formulas described above are called **canonical sentences**. Fix a natural number n and construct a set of canonical sentences, denoted $\mathcal{C}_{h,n}$, by induction on h. Let $\mathcal{C}_{0,n} = \{\widehat{T} \mid T \subseteq \{p_1, \ldots, p_n\}\}$. Suppose that $\mathcal{C}_{h,n}$ has been defined and that $S \subseteq \mathcal{C}_{h,n}$ and $T \subseteq \{p_1, \ldots, p_n\}$. Define the formula

$$\alpha_{S,T} := \bigwedge_{\psi \in S} \Diamond\psi \wedge \Box \bigvee S \wedge \widehat{T}$$

and let $\mathcal{C}_{h+1,n} = \{\alpha_{S,T} \mid S \subseteq \mathcal{C}_{h,n},\ T \subseteq \{p_1, \ldots, p_n\}\}$. It is not hard to see that formulas of the form $\alpha_{S,T}$ play the same role in modal logic as the formulas $\widehat{T}$ in propositional logic. That is, $\alpha_{S,T}$ can be thought of as a complete description of a modal state of affairs. This is justified by the following Lemma from [1]. The proof can be found in [1] although we will repeat it here in the interest of exposition.

Lemma 3.1 *For any modal formula φ of modal depth at most h built from propositional variables $\{p_1, \ldots, p_n\}$ and any $\alpha_{S,T} \in \mathcal{C}_{h+1,n}$ exactly one of the following holds $\vdash \alpha_{S,T} \to \varphi$ or $\vdash \alpha_{S,T} \to \neg\varphi$.*

Proof. The proof is by induction on φ. The base case is obvious as are the boolean connectives. We consider only the modal case. Suppose that statement holds for ψ and consider the formula $\Diamond\psi$. Note that for each $\beta \in S$, the induction hypothesis applies to β and ψ. Thus for each $\beta \in S$, either $\vdash \beta \to \psi$ or $\vdash \beta \to \neg\psi$. There are two cases: 1. there is some $\beta \in S$ such that $\vdash \beta \to \psi$ and 2. for each $\beta \in S$, $\vdash \beta \to \neg\psi$. Suppose case 1 holds and $\beta \in S$ is such that $\vdash \beta \to \psi$. Then, it is easy to show that in **K**, $\vdash \Diamond\beta \to \Diamond\psi$. Hence, by construction of $\alpha_{S,T}$, $\vdash \alpha_{S,T} \to \Diamond\psi$. Suppose we are in the second case. Using propositional reasoning, $\vdash \bigvee S \to \neg\psi$. Then, $\vdash \Box\bigvee S \to \Box\neg\psi$. Hence, by construction of $\alpha_{S,T}$, $\vdash \alpha_{S,T} \to \neg\Diamond\psi$. QED

This lemma demonstrates that we can think of these formulas as complete descriptions of a state (up to finite depth) in some Kripke structure. There are a few other facts that are relevant at this point. The proofs can be found in [1] and we will not repeat them here. Given a set of formulas X, let $\bigoplus X$ denote *exactly one of* X. Formally, if $X = \{\varphi_1, \ldots, \varphi_n\}$, then $\bigoplus X$ is short for $\bigvee_{i=1,\ldots,n}(\varphi_i \wedge \neg \bigvee_{j\neq i} \varphi_j)$.

Lemma 3.2 *1. For any h, $\vdash \bigoplus \mathcal{C}_{h,n}$ (and hence $\vdash \bigvee \mathcal{C}_{h,n}$)*

2. For any formula φ of height h, $\vdash \varphi \leftrightarrow \bigvee\{\alpha \mid \alpha \in \mathcal{C}_{h,n},\ \vdash \alpha \to \varphi\}$

Moss constructs a (finite) Kripke model from the set of formulas $\mathcal{C}_{h,n}$ as follows. Let $\mathbb{C}_{h,n} = \langle \mathcal{C}, R, V\rangle$ where

1. $\mathcal{C} \subseteq \mathcal{C}_{h,n}$ is the set of all **K-consistent** formulas from $\mathcal{C}_{h,n}$
2. For $\alpha, \beta \in \mathcal{C}$, $\alpha R \beta$ provided $\alpha \wedge \Diamond\beta$ is consistent
3. for $p \in \{p_1, \ldots, p_n\}$, $V(p) = \{\alpha \mid \alpha \in \mathcal{C}, \vdash \alpha \to p\}$.

The truth Lemma connects truth of φ at a state α and the derivability of the implication $\alpha \to \varphi$. We first need an existence Lemma whose proof can be found in [1]

Lemma 3.3 (Existence Lemma, [1]) *Suppose that $\varphi \in \mathcal{L}_{h,n}$ and $\mathbb{C}_{h,n} = \langle \mathcal{C}, R, V\rangle$ is as defined above. If $\alpha \wedge \Diamond\varphi$ is* **K***-consistent then there is a $\beta \in \mathcal{C}$ such that $\alpha \wedge \Diamond\beta$ is* **K***-consistent and $\vdash \beta \to \varphi$.*

The proof uses Lemma 3.2 and can be found in [1].

Lemma 3.4 (Truth Lemma, [1]) *Suppose that $\varphi \in \mathcal{L}_{h,n}$ and $\mathbb{C}_{h,n} = \langle \mathcal{C}, R, V\rangle$ is as defined above. Then for each $\alpha \in \mathcal{C}$, $\mathbb{C}_{h,n}, \alpha \models \varphi$ iff $\vdash_{\mathbf{K}} \alpha \to \varphi$.*

Proof. As usual, the proof is by induction on φ. The base case and boolean connectives are straightforward. The only interesting case is the modal operator. Suppose that $\mathbb{C}_{h,n}, \alpha \models \Diamond\psi$. Then there is some $\beta \in \mathcal{C}$ such that $\alpha R \beta$ and $\mathbb{C}_{h,n}, \beta \models \psi$. By the definition of R, $\alpha \wedge \Diamond\beta$ is **K**-consistent. By Lemma 3.1, either $\vdash \alpha \to \Diamond\psi$ or $\vdash \alpha \to \neg\Diamond\psi$. If $\vdash \alpha \to \Diamond\psi$ we are done. Suppose that $\vdash \alpha \to \neg\Diamond\psi$. Now, by the induction hypothesis, $\vdash \beta \to \psi$. Hence $\vdash \Diamond\beta \to \Diamond\psi$. But this contradicts the assumption that $\alpha \wedge \Diamond\beta$ is **K**-consistent. Suppose that $\vdash \alpha \to \Diamond\psi$. Then $\alpha \wedge \Diamond\psi$ is **K**-consistent. Hence by Lemma 3.3, there is a $\beta \in \mathcal{C}$ such that $\alpha \wedge \Diamond\beta$ is **K**-consistent and $\vdash \beta \to \psi$. But this means that $\mathbb{C}_{h,n}, \alpha \models \Diamond\psi$. QED

The weak completeness theorem easily follows from the above Lemmas.

Theorem 3.5 **K** *is weakly complete, i.e., for each $\varphi \in \mathcal{L}_\Diamond$, if $\models \varphi$, then $\vdash_{\mathbf{K}} \varphi$.*

Proof. Let h and n be large enough so that $\varphi \in \mathcal{L}_{h,n}$ and suppose that $\models \varphi$. Then, in particular, φ is valid in $\mathbb{C}_{h,n}$. Thus for each $\alpha \in \mathcal{C}$, $\mathbb{C}_{h,n}, \alpha \models \varphi$. Hence by Lemma 3.4, for each $\alpha \in \mathcal{C}$, $\vdash \alpha \rightarrow \varphi$. Hence, $\vdash \bigvee \mathcal{C} \rightarrow \varphi$. By Lemma 3.2, $\vdash \bigvee \mathcal{C}$. Therefore, $\vdash \varphi$. QED

In [1], Moss uses the above technique to show that a number of well-known modal logics are weakly complete.

References

[1] Larry Moss Finite models constructed from canonical formulas. Journal of Philosophical Logic, 36:6, pp. 605 - 640, 2005.