Modal Logic
Common Knowledge and Agreement Theorems

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Plan

- Common Knowledge
- Agreeing to Disagree

"*Common Knowledge*" is informally described as what any fool would know, given a certain situation: It encompasses what is relevant, agreed upon, established by precedent, assumed, being attended to, salient, or in the conversational record.

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It is not Common Knowledge who "defined" Common Knowledge!

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Shared situation: There is a *shared situation* s such that (1) s entails φ, (2) s entails everyone knows φ, plus other conditions

H. Clark and C. Marshall. *Definite Reference and Mutual Knowledge*. 1981.

M. Gilbert. *On Social Facts*. Princeton University Press (1989).

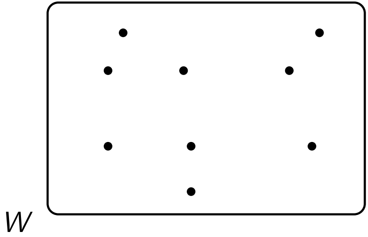
P. Vanderschraaf and G. Sillari. *"Common Knowledge", The Stanford Encyclopedia of Philosophy (2009)*.
http://plato.stanford.edu/entries/common-knowledge/.

The "Standard" Account

R. Aumann. *Agreeing to Disagree*. Annals of Statistics (1976).

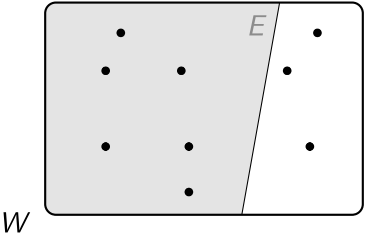
R. Fagin, J. Halpern, Y. Moses and M. Vardi. *Reasoning about Knowledge*. MIT Press, 1995.

The “Standard” Account



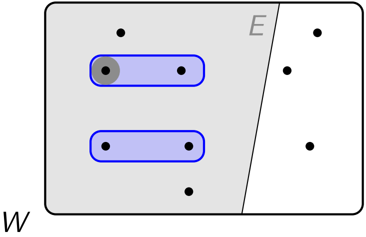
W is a set of **states** or **worlds**.

The "Standard" Account



An **event**/**proposition** is any (definable) subset $E \subseteq W$

The “Standard” Account



At each state, agents are assigned a set of states they *consider possible* (according to their information).
The information may be (in)correct, partitional, ...

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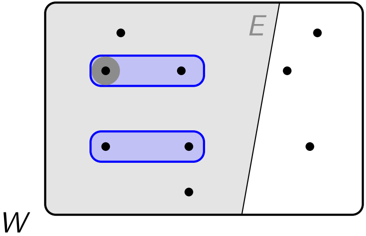
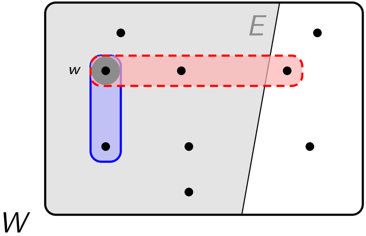


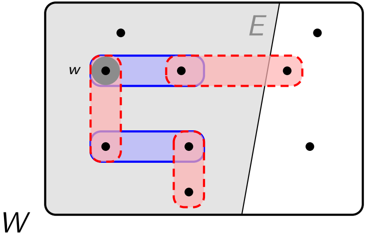
Knowledge Function: $K_i : \wp(W) \rightarrow \wp(W)$ where $K_i(E) = \{w \mid R_i(w) \subseteq E\}$

The “Standard” Account



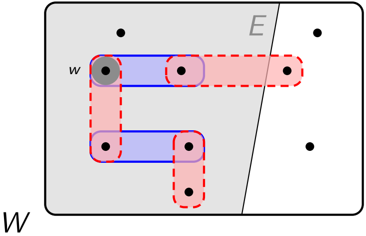
$w \in K_A(E)$ and $w \notin K_B(E)$

The “Standard” Account



The model also describes the agents’ **higher-order knowledge/beliefs**

The “Standard” Account



Everyone Knows: $K(E) = \bigcap_{i\in\mathcal{A}} K_i(E)$, $K^0(E) = E$, $K^m(E) = K(K^{m-1}(E))$

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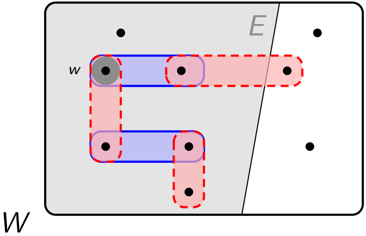
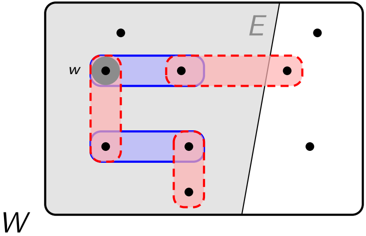


Common Knowledge: $C : \wp(W) \to \wp(W)$ with

$$C(E) = \bigcap_{m \geq 0} K^m(E)$$

The "Standard" Account



$w \in K(E)$ $\qquad$ $w \notin C(E)$

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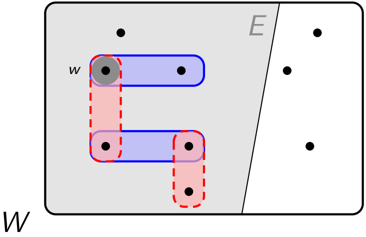

$w \in C(E)$

Fact. For all $i \in \mathcal{A}$ and $E \subseteq W$, $K_i C(E) = C(E)$.

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> Suppose you are told "Ann and Bob are going together,"' and respond "sure, that's common knowledge." What you mean is not only that everyone knows this, but also that the announcement is pointless, occasions no surprise, reveals nothing new; in effect, that the situation after the announcement does not differ from that before. ...the event "Ann and Bob are going together" — call it E — is common knowledge if and only if some event — call it F — happened that entails E and also entails all players' knowing F (like all players met Ann and Bob at an intimate party). *(Aumann, pg. 271, footnote 8)*

Fact. For all $i \in \mathcal{A}$ and $E \subseteq W$, $K_i C(E) = C(E)$.

An event F is **self-evident** if $K_i(F) = F$ for all $i \in \mathcal{A}$.

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Fact. $w \in C(E)$ if every finite path starting at w ends in a state in E

The following axiomatize common knowledge:

- $C(\varphi \to \psi) \to (C\varphi \to C\psi)$
- $C\varphi \to (\varphi \wedge EC\varphi)$ (Fixed-Point)
- $C(\varphi \to E\varphi) \to (\varphi \to C\varphi)$ (Induction)

An Example

Two players Ann and Bob are told that the following will happen. Some positive integer n will be chosen and *one* of n, $n + 1$ will be written on Ann's forehead, the other on Bob's. Each will be able to see the other's forehead, but not his/her own.

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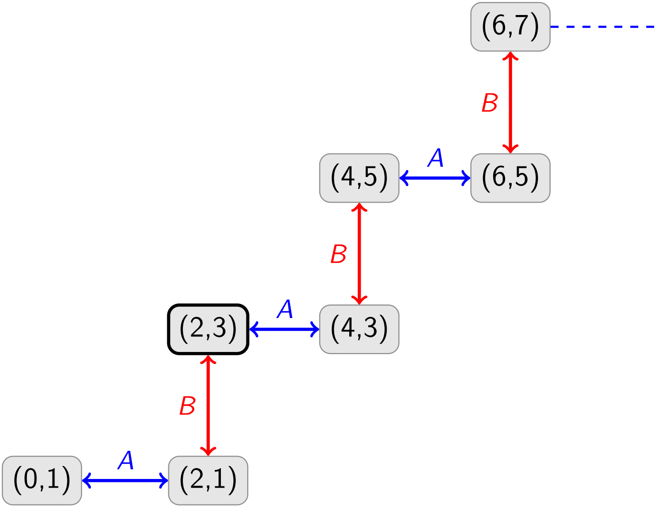
Do the agents know there numbers are less than 1000?

Is it common knowledge that their numbers are less than 1000?


(6,7)
B
(4,5)
A
(6,5)
B
(2,3)
A
(4,3)
B
(0,1)
A
(2,1)

Agreeing to Disagree

Theorem: Suppose that n agents share a common prior and have different private information. If there is common knowledge in the group of the posterior probabilities, then the posteriors must be equal.

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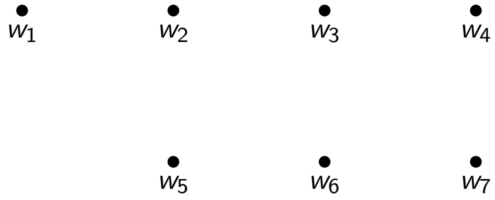
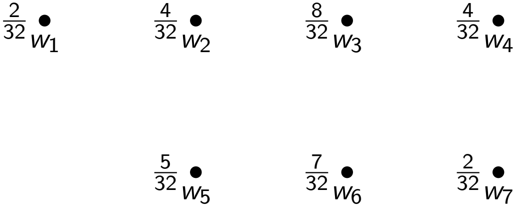
2 Scientists Perform an Experiment



They agree the true state is one of seven different states.

2 Scientists Perform an Experiment



They agree on a common prior.

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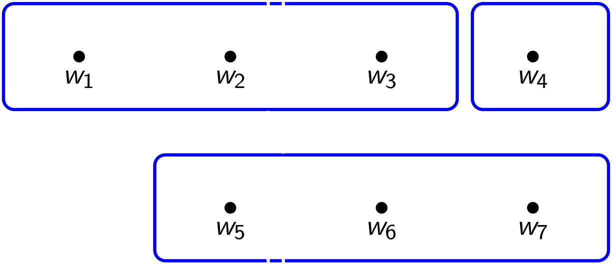

They agree that Experiment 1 would produce the blue partition.

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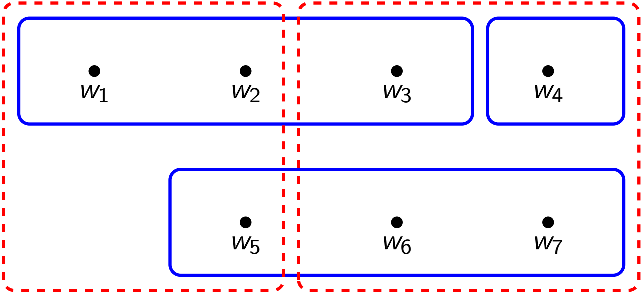
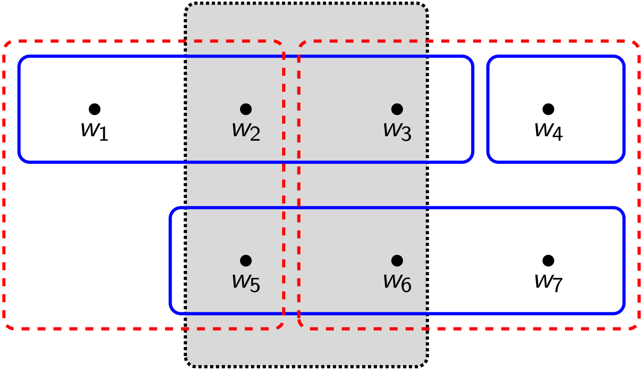


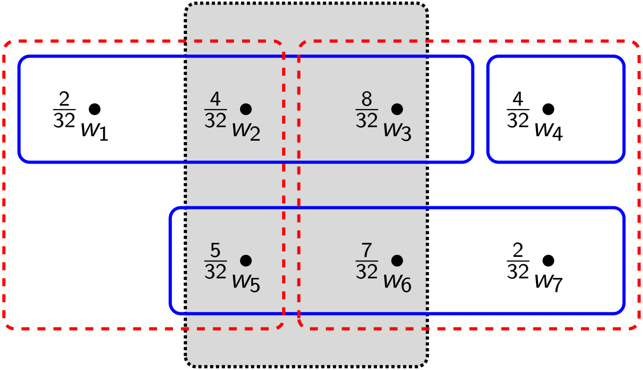
They agree that Experiment 1 would produce the blue partition
and Experiment 2 the red partition.

2 Scientists Perform an Experiment



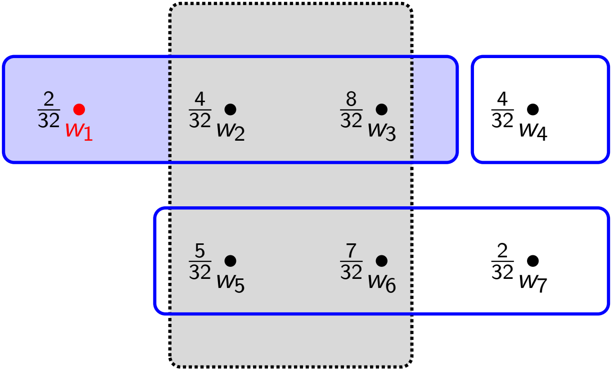
They are interested in the truth of $E = \{w_2, w_3, w_5, w_6\}$.

2 Scientists Perform an Experiment



So, they agree that $P(E) = \frac{24}{32}$.

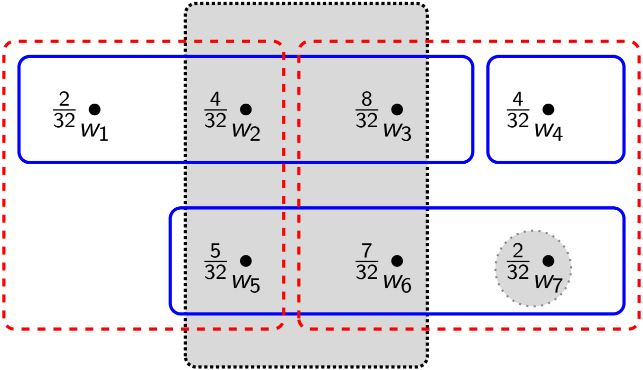
2 Scientists Perform an Experiment



Also, that *if the true state is* w_1, then Experiment 1 will yield

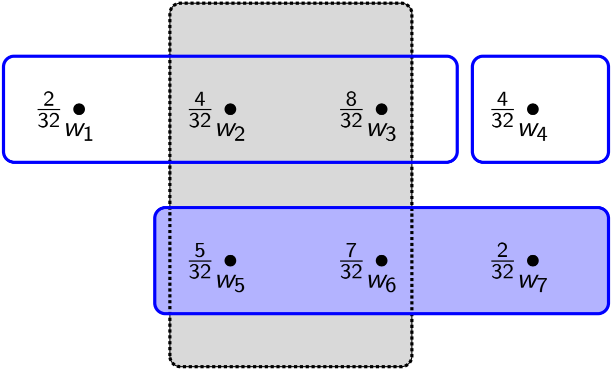
$$P(E|I) = \frac{P(E \cap I)}{P(I)} = \frac{12}{14}$$

2 Scientists Perform an Experiment



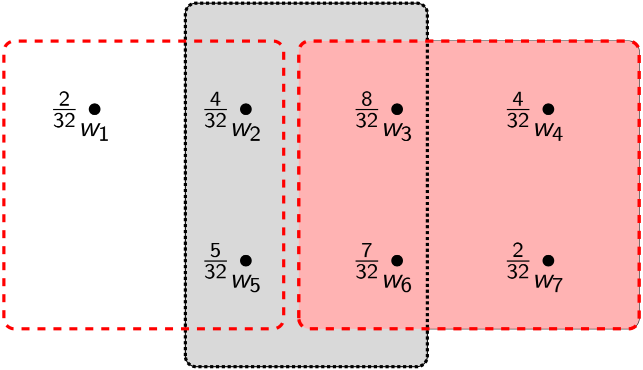
Suppose the true state is w_7 and the agents preform the experiments.

2 Scientists Perform an Experiment



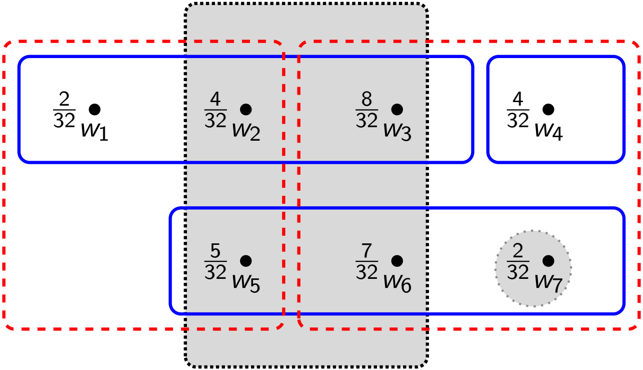
Suppose the true state is w_7, then $Pr_1(E) = \frac{12}{14}$

2 Scientists Perform an Experiment



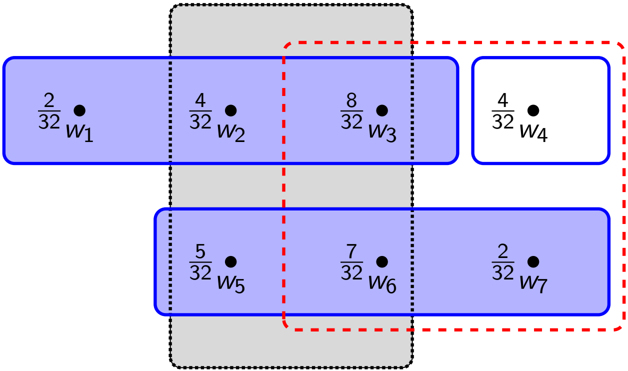
Then $Pr_1(E) = \frac{12}{14}$ and $Pr_2(E) = \frac{15}{21}$

2 Scientists Perform an Experiment



Suppose they exchange emails with the new subjective probabilities: $Pr_1(E) = \frac{12}{14}$ and $Pr_2(E) = \frac{15}{21}$

2 Scientists Perform an Experiment



Agent 2 learns that w_4 is **NOT** the true state (same for Agent 1).

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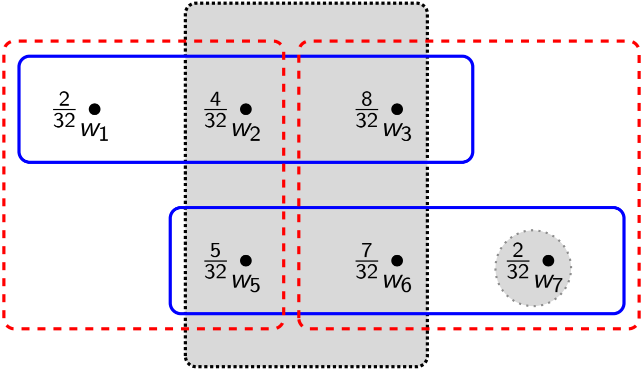

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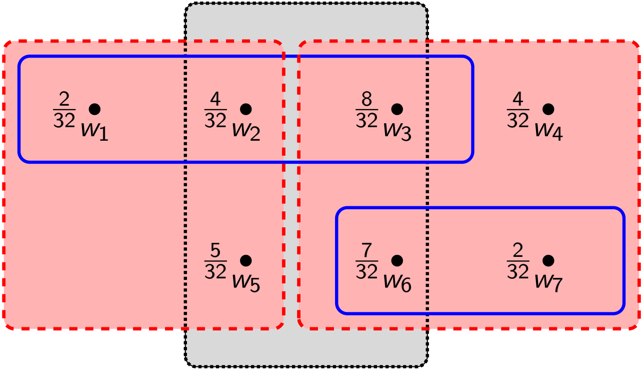

Agent 1 learns that w_5 is **NOT** the true state (same for Agent 1).

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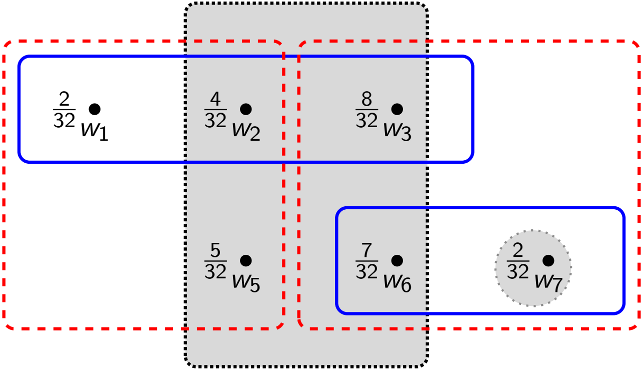
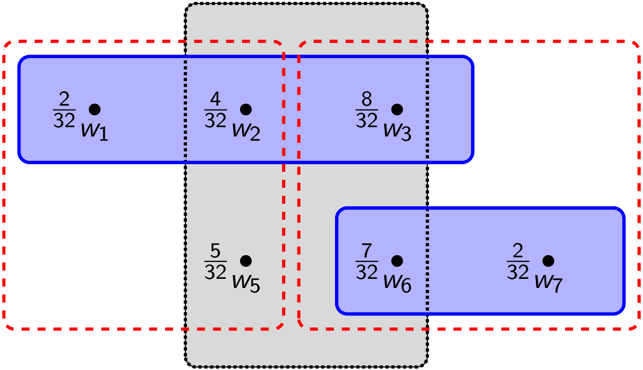


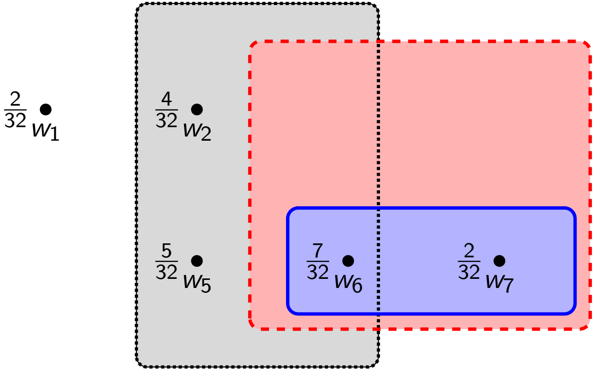
The new probabilities are $Pr_1(E|I') = \frac{7}{9}$ and $Pr_2(E|I') = \frac{15}{17}$

2 Scientists Perform an Experiment



After exchanging this information ($Pr_1(E|I') = \frac{7}{9}$ and $Pr_2(E|I') = \frac{15}{17}$), Agent 2 learns that w_3 is **NOT** the true state.

2 Scientists Perform an Experiment



No more revisions are possible and the agents agree on the posterior probabilities.

Dissecting Aumann's Theorem

- Qualitative versions: like-minded individuals cannot agree to make different decisions.

M. Bacharach. *Some Extensions of a Claim of Aumann in an Axiomatic Model of Knowledge*. Journal of Economic Theory (1985).

J.A.K. Cave. *Learning to Agree*. Economic Letters (1983).

D. Samet. *The Sure-Thing Principle and Independence of Irrelevant Knowledge*. 2008.

D. Samet. *Agreeing to disagree: The non-probabilistic case*. Games and Economic Behavior, Vol. 69, 2010, 169-174.

The Framework

Knowledge Structure: $\langle W, \{\Pi_i\}_{i \in \mathcal{A}} \rangle$ where each Π_i is a partition on W ($\Pi_i(w)$ is the cell in Π_i containing w).

Decision Function: Let D be a nonempty set of **decisions**. A decision function for $i \in \mathcal{A}$ is a function $\mathbf{d}_i : W \to D$. A vector $\mathbf{d} = (\mathbf{d}_1, \ldots, \mathbf{d}_n)$ is a decision function profile. Let $[\mathbf{d}_i = d] = \{w \mid \mathbf{d}_i(w) = d\}$.

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(A1) Each agent knows her own decision:

$$[\mathbf{d}_i = d] \subseteq K_i([\mathbf{d}_i = d])$$

Comparing Knowledge

$[j \succeq i]$: agent j is at least as knowledgeable as agent i.

$$[j \succeq i] := \bigcap_{E \in \wp(W)} (K_i(E) \Rightarrow K_j(E)) = \bigcap_{E \in \wp(W)} (\neg K_i(E) \cup K_j(E))$$

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$[j \sim i] = [j \succeq i] \cap [i \succeq j]$

Interpersonal Sure-Thing Principle (ISTP)

For any pair of agents i and j and decision d,

$$K_i([j \succeq i] \cap [\mathbf{d}_j = d]) \subseteq [\mathbf{d}_i = d]$$

Interpersonal Sure-Thing Principle (ISTP): Illustration

Suppose that Alice and Bob, two detectives who graduated the same police academy, are assigned to investigate a murder case.

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Interpersonal Sure-Thing Principle (ISTP): Illustration

Suppose that Alice and Bob, two detectives who graduated the same police academy, are assigned to investigate a murder case. If they are exposed to different evidence, they may reach different decisions. Yet, being the students of the same academy, the method by which they arrive at their conclusions is the same. Suppose now that detective Bob, a father of four who returns home every day at five oclock, collects all the information about the case at hand together with detective Alice.

Interpersonal Sure-Thing Principle (ISTP): Illustration

However, Alice, single and a workaholic, continues to collect more information every day until the wee hours of the morning — information which she does not necessarily share with Bob.

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Interpersonal Sure-Thing Principle (ISTP): Illustration

However, Alice, single and a workaholic, continues to collect more information every day until the wee hours of the morning — information which she does not necessarily share with Bob. Obviously, Bob knows that Alice is at least as knowledgeable as he is. Suppose that he also knows what Alices decision is. Since Alice uses the same investigation method as Bob, he knows that had he been in possession of the more extensive knowledge that Alice has collected, he would have made the same decision as she did. Thus, this is indeed his decision.

Implications of ISTP

Proposition. If the decision function profile $\mathbf{d}$ satisfies ISTP, then

$$[i \sim j] \subseteq \bigcup_{d \in D} ([\mathbf{d}_i = d] \cap [\mathbf{d}_j = d])$$

ISTP Expandability

Agent i is an **epistemic dummy** if it is always the case that all the agents are at least as knowledgeable as i. That is, for each agent j,

$$[j \succeq i] = W$$

A decision function profile $\mathbf{d}$ on $\langle W, \Pi_1, \ldots, \Pi_n \rangle$ is **ISTP expandable** if for any expanded structure $\langle W, \Pi_1, \ldots, \Pi_{n+1} \rangle$ where $n + 1$ is an epistemic dummy, there exists a decision function $\mathbf{d}_{n+1}$ such that $(\mathbf{d}_1, \mathbf{d}_2, \ldots, \mathbf{d}_{n+1})$ satisfies ISTP.

ISTP Expandability: Illustration

Suppose that after making their decisions, Alice and Bob are told that another detective, one E.P. Dummy, who graduated the very same police academy, had also been assigned to investigate the same case.

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ISTP Expandability: Illustration

But this is not so in the case of detective Dummy. It is commonly known that the only information source of this detective, known among his colleagues as the couch detective, is the TV set.

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Generalized Agreement Theorem

If **d** is an ISTP expandable decision function profile on a partition structure $\langle W, \Pi_1, \ldots, \Pi_n \rangle$, then for any decisions $d_1, \ldots, d_n$ which are not identical, $C(\bigcap_i [\mathbf{d}_i = d_i]) = \emptyset$.

Common p-belief

Theorem. If the posteriors of an event X are common p-belief at some state w, then any two posteriors can differ by at most $2(1-p)$.

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Next: Dynamics of Knowledge and Belief