Neighborhood Semantics for Modal Logic

Lecture 4

Eric Pacuit

ILLC, Universiteit van Amsterdam
staff.science.uva.nl/~epacuit

August 16, 2007

✓ Introduction, Motivation and Background Information

✓ Basic Concepts, Non-normal Modal Logics, Completeness, Incompleteness, Relation with Relational Semantics

✓ Decidability/Complexity, Topological Semantics for Modal Logic,

Lecture 4: Advanced Topics — Topological Semantics for Modal Logic, some Model Theory

Lecture 5: Neighborhood Semantics in Action: Game Logic, Coalgebra, Common Knowledge, First-Order Modal Logic

Sketch of Completeness of First-Order Modal Logic

Theorem $\mathbf{FOL} + \mathbf{E} + CBF$ is sound and strongly complete with respect to the class of frames that are either non-trivial and supplemented or trivial and not supplemented.

Sketch of Completeness of First-Order Modal Logic

Theorem $\mathbf{FOL} + \mathbf{E} + CBF$ is sound and strongly complete with respect to the class of frames that are either non-trivial and supplemented or trivial and not supplemented.

Theorem $\mathbf{FOL} + \mathbf{K}$ is sound and strongly complete with respect to the class of filters.

Lemma The augmentation of the smallest canonical model for $\mathbf{FOL} + \mathbf{K} + BF$ is a canonical for $\mathbf{FOL} + \mathbf{K} + BF$.

Theorem $\mathbf{FOL} + \mathbf{K} + BF$ is sound and strongly complete with respect to the class of augmented first-order neighborhood frames.

What is the relationship between Neighborhood and other Semantics for Modal Logic? What about First-Order Modal Logic?

Can we import results/ideas from model theory for modal logic with respect to Kripke Semantics/Topological Semantics?

Model Constructions

- Disjoint Union
- Generated Submodel
- Bounded Morphism
- Bisimulation

Bounded Morphism

Let $\mathfrak{F}_1 = \langle W_1, N_1 \rangle$ and $\mathfrak{F}_2 = \langle W_2, N_2 \rangle$ be two neighbourhood frames.

A bounded morphism from $\mathfrak{F}_1$ to $\mathfrak{F}_2$ is a map $f : W_1 \rightarrow W_2$ such that

$$\text{for all } X \subseteq W_2,\ f^{-1}[X] \in N_1(w) \text{ iff } X \in N_2(f(w))$$

Bounded Morphism

Let $\mathfrak{F}_1 = \langle W_1, N_1 \rangle$ and $\mathfrak{F}_2 = \langle W_2, N_2 \rangle$ be two neighbourhood frames.

A bounded morphism from $\mathfrak{F}_1$ to $\mathfrak{F}_2$ is a map $f : W_1 \to W_2$ such that

$$\text{for all } X \subseteq W_2,\ f^{-1}[X] \in N_1(w) \text{ iff } X \in N_2(f(w))$$

If $\mathfrak{M}_1 = \langle W_1, N_1, V_1 \rangle$ and $\mathfrak{M}_2 = \langle W_2, N_2, V_2 \rangle$ if f is a bounded morphism from $\langle W_1, N_1 \rangle$ to $\langle W_2, N_2 \rangle$ and for all p, $w \in V_1(p)$ iff $f(w) \in V_2(p)$.

Bounded Morphism

Let $\mathfrak{F}_1 = \langle W_1, N_1 \rangle$ and $\mathfrak{F}_2 = \langle W_2, N_2 \rangle$ be two neighbourhood frames.

A bounded morphism from $\mathfrak{F}_1$ to $\mathfrak{F}_2$ is a map $f : W_1 \to W_2$ such that

$$\text{for all } X \subseteq W_2,\ f^{-1}[X] \in N_1(w) \text{ iff } X \in N_2(f(w))$$

Lemma

Let $\mathfrak{M}_1 = \langle W_1, N_1, V_1 \rangle$ and $\mathfrak{M}_2 = \langle W_2, N_2, V_2 \rangle$ be two neighbourhood models and $f : W_1 \to W_2$ a bounded morphism. Then for each modal formula $\varphi \in \mathcal{L}$ and state $w \in W_1$, $\mathfrak{M}_1, w \models \varphi$ iff $\mathfrak{M}_2, f(w) \models \varphi$.

Disjoint Union

Let $\mathfrak{M}_1 = \langle W_1, N_1, V_1 \rangle$ and $\mathfrak{M}_2 = \langle W_2, N_2, V_2 \rangle$ with $W_1 \cap W_2 = \emptyset$ be two neighborhood models.

Disjoint Union

Let $\mathfrak{M}_1 = \langle W_1, N_1, V_1 \rangle$ and $\mathfrak{M}_2 = \langle W_2, N_2, V_2 \rangle$ with $W_1 \cap W_2 = \emptyset$ be two neighborhood models.

The disjoint union is the structure $\mathfrak{M}_1 + \mathfrak{M}_2 = \langle W, N, V \rangle$ where

Disjoint Union

Let $\mathfrak{M}_1 = \langle W_1, N_1, V_1 \rangle$ and $\mathfrak{M}_2 = \langle W_2, N_2, V_2 \rangle$ with $W_1 \cap W_2 = \emptyset$ be two neighborhood models.

The disjoint union is the structure $\mathfrak{M}_1 + \mathfrak{M}_2 = \langle W, N, V \rangle$ where

- $W = W_1 \cup S_2$,

Disjoint Union

Let $\mathfrak{M}_1 = \langle W_1, N_1, V_1 \rangle$ and $\mathfrak{M}_2 = \langle W_2, N_2, V_2 \rangle$ with $W_1 \cap W_2 = \emptyset$ be two neighborhood models.

The disjoint union is the structure $\mathfrak{M}_1 + \mathfrak{M}_2 = \langle W, N, V \rangle$ where

- $W = W_1 \cup S_2$,
- for each $p \in \text{At}$, $V(p) = V_1(p) \cup V_2(p)$ and

Disjoint Union

Let $\mathfrak{M}_1 = \langle W_1, N_1, V_1 \rangle$ and $\mathfrak{M}_2 = \langle W_2, N_2, V_2 \rangle$ with $W_1 \cap W_2 = \emptyset$ be two neighborhood models.

The disjoint union is the structure $\mathfrak{M}_1 + \mathfrak{M}_2 = \langle W, N, V \rangle$ where

- $W = W_1 \cup S_2$,
- for each $p \in \text{At}$, $V(p) = V_1(p) \cup V_2(p)$ and
- For each $X \subseteq W$ and $w \in W_i$,
  $X \in N(w)$ iff $X \cap W_i \in N_i(w)$, $(i = 1, 2)$

Disjoint Union

Let $\mathfrak{M}_1 = \langle W_1, N_1, V_1 \rangle$ and $\mathfrak{M}_2 = \langle W_2, N_2, V_2 \rangle$ with $W_1 \cap W_2 = \emptyset$ be two neighborhood models.

The disjoint union is the structure $\mathfrak{M}_1 + \mathfrak{M}_2 = \langle W, N, V \rangle$ where

- $W = W_1 \cup S_2$,
- for each $p \in \mathrm{At}$, $V(p) = V_1(p) \cup V_2(p)$ and
- For each $X \subseteq W$ and $w \in W_i$,
  $X \in N(w)$ iff $X \cap W_i \in N_i(w)$, $(i = 1, 2)$

Lemma

For each collection of Neighborhood models $\{\mathfrak{M}_i \mid i \in I\}$, for each $w \in W_i$, $\mathfrak{M}_i, w \models \varphi$ iff $\biguplus_{i \in I} \mathfrak{M}_i, w \models \varphi$

Generated Submodels?

Bisimulations?

Tree Model Property?

First-Order Correspondence Language?

A neighborhood frame is monotonic if $N(w)$ is closed under supersets.

H. Hansen. *Monotonic Modal Logic*. 2003.

Alternative Characterization of Bounded Morphism

$f : \mathfrak{M}_1 = \langle W_1, N_1, V_1 \rangle \rightarrow \mathfrak{M}_2 \langle W_2, N_2, V_2 \rangle$ is a bounded morphism iff

Alternative Characterization of Bounded Morphism

$f : \mathfrak{M}_1 = \langle W_1, N_1, V_1 \rangle \rightarrow \mathfrak{M}_2 \langle W_2, N_2, V_2 \rangle$ is a bounded morphism iff

Atomic harmony: for each $p \in \mathrm{At}$, $w \in V_1(p)$ iff $f(w) \in V_2(p)$

Alternative Characterization of Bounded Morphism

$f : \mathfrak{M}_1 = \langle W_1, N_1, V_1 \rangle \to \mathfrak{M}_2 \langle W_2, N_2, V_2 \rangle$ is a bounded morphism iff

Atomic harmony: for each $p \in \text{At}$, $w \in V_1(p)$ iff $f(w) \in V_2(p)$

Morphism: If $X \in N_1(w)$ then $f[X] \in N_2(f(w))$

Alternative Characterization of Bounded Morphism

$f : \mathfrak{M}_1 = \langle W_1, N_1, V_1 \rangle \rightarrow \mathfrak{M}_2 \langle W_2, N_2, V_2 \rangle$ is a bounded morphism iff

Atomic harmony: for each $p \in \text{At}$, $w \in V_1(p)$ iff $f(w) \in V_2(p)$

Morphism: If $X \in N_1(w)$ then $f[X] \in N_2(f(w))$

Zag: If $X' \in N_2(f(w))$ then there is an $X \subseteq W$ such that

$$f[X] \subseteq X' \text{ and } X \in N_1(w)$$

Generated Submodel

Let $\mathfrak{M}' = \langle W', N', V' \rangle$ be a submodel of $\mathfrak{M} = \langle W, N, V \rangle$. We want:

Generated Submodel

Let $\mathfrak{M}' = \langle W', N', V' \rangle$ be a submodel of $\mathfrak{M} = \langle W, N, V \rangle$. We want:

$\mathfrak{M}'$ is a generated submodel if

$$\text{If } w' \in W' \text{ and } X' \in N(w') \text{ then } X' \subseteq W'$$

Generated Submodel

Let $\mathfrak{M}' = \langle W', N', V' \rangle$ be a submodel of $\mathfrak{M} = \langle W, N, V \rangle$. We want:

$\mathfrak{M}'$ is a generated submodel if

$$\text{If } w' \in W' \text{ and } X' \in N(w') \text{ then } X' \subseteq W'$$

But then, there are only 2 generated submodels!

Generated Submodel

Let $\mathfrak{M}' = \langle W', N', V' \rangle$ be a submodel of $\mathfrak{M} = \langle W, N, V \rangle$. We want:

$\mathfrak{M}'$ is a generated submodel if the identity map $i : W' \to W$ is a bounded morphism: for all $w' \in W'$ and $X \subseteq W$

$$i^{-1}[X] = X \cap W' \in N'(w') \text{ iff } X \in N(w')$$

Generated Submodel

Let $\mathfrak{M}' = \langle W', N', V' \rangle$ be a submodel of $\mathfrak{M} = \langle W, N, V \rangle$. We want:

$\mathfrak{M}'$ is a generated submodel if the identity map $i : W' \to W$ is a bounded morphism: for all $w' \in W'$ and $X \subseteq W$

$$i^{-1}[X] = X \cap W' \in N'(w') \text{ iff } X \in N(w')$$

Lemma

Let $\mathfrak{M}' = \langle W', N', V' \rangle$ be a generated submodel of $\mathfrak{M} = \langle W, N, V \rangle$. Then for all $\varphi \in \mathcal{L}$ and $w \in W'$,
$\mathfrak{M}', w \models \varphi$ iff $\mathfrak{M}, w \models \varphi$

Generated Submodel

Let $\mathfrak{M}' = \langle W', N', V' \rangle$ be a submodel of $\mathfrak{M} = \langle W, N, V \rangle$. We want:

$\mathfrak{M}'$ is a generated submodel if the identity map $i : W' \to W$ is a bounded morphism: for all $w' \in W'$ and $X \subseteq W$

$$i^{-1}[X] = X \cap W' \in N'(w') \text{ iff } X \in N(w')$$

Lemma

If f is an injective bounded morphism from $\mathfrak{M}' = \langle W', N', V' \rangle$ to $\mathfrak{M} = \langle W, N, V \rangle$, then $\mathfrak{M}'|_{f[W']}$ is a generated submodel of $\mathfrak{M}'$.

Bisimulation

Let $\mathfrak{M} = \langle W, N, V\rangle$ and $\mathfrak{M}' = \langle W', N', V'\rangle$ be two neighborhood models. A relation $Z \subseteq W \times W'$ is a bisimulation provided whenever wZw':

Atomic harmony: for each $p \in \mathsf{At}$, $w \in V(p)$ iff $w' \in V'(p)$

Bisimulation

Let $\mathfrak{M} = \langle W, N, V \rangle$ and $\mathfrak{M}' = \langle W', N', V' \rangle$ be two neighborhood models. A relation $Z \subseteq W \times W'$ is a bisimulation provided whenever wZw':

Atomic harmony: for each $p \in \text{At}$, $w \in V(p)$ iff $w' \in V'(p)$

Zig: If $X \in N(w)$ then there is an $X' \subseteq W'$ such that

$$X' \in N'(w') \text{ and } \forall x' \in X' \; \exists x \in X \text{ such that } xZx'$$

Bisimulation

Let $\mathfrak{M} = \langle W, N, V \rangle$ and $\mathfrak{M}' = \langle W', N', V' \rangle$ be two neighborhood models. A relation $Z \subseteq W \times W'$ is a bisimulation provided whenever wZw':

Atomic harmony: for each $p \in \text{At}$, $w \in V(p)$ iff $w' \in V'(p)$

Zig: If $X \in N(w)$ then there is an $X' \subseteq W'$ such that

$$X' \in N'(w') \text{ and } \forall x' \in X' \; \exists x \in X \text{ such that } xZx'$$

Zag: If $X' \in N'(w')$ then there is an $X \subseteq W$ such that

$$X \in N(w) \text{ and } \forall x \in X \; \exists x' \in X' \text{ such that } xZx'$$

Bisimulation

Let $\mathfrak{M} = \langle W, N, V \rangle$ and $\mathfrak{M}' = \langle W', N', V' \rangle$ be two neighborhood models. A relation $Z \subseteq W \times W'$ is a bisimulation provided whenever wZw':

Atomic harmony: for each $p \in \text{At}$, $w \in V(p)$ iff $w' \in V'(p)$

Zig: If $X \in N(w)$ then there is an $X' \subseteq W'$ such that

$$X' \in N'(w') \text{ and } \forall x' \in X' \; \exists x \in X \text{ such that } xZx'$$

Zag: If $X' \in N'(w')$ then there is an $X \subseteq W$ such that

$$X \in N(w) \text{ and } \forall x \in X \; \exists x' \in X' \text{ such that } xZx'$$

Lemma

If $\mathbb{M}, w \underline{\leftrightarrow} \mathbb{M}', w'$ *then* $\mathbb{M}, w \leftrightsquigarrow \mathbb{M}', w'$.

Non-monotonic Core

Let $\langle W, N\rangle$ be a monotonic frame. The non-monotonic core of N, denote N^c is defined as follows:

$$X \in N^c(w) \text{ iff } X \in N(w) \text{ and for all } X' \subsetneq X,\ X' \notin N(w)$$

A monotonic model is core complete provided for each $X \subseteq W$, if $X \in N(w)$ then there is a $C \in N^c(w)$ such that $C \subseteq X$.

Restricting to the non-monotonic core

f is a bounded core morphism from $\mathfrak{M}_1$ to $\mathfrak{M}_2$ provided:

Atomic harmony: for each $p \in \mathsf{At}$, $w \in V(p)$ iff $f(w) \in V'(p)$

Morphism: If $X \in N_1^c(w)$ then $f[X] \in N_2^c(f(w))$

Zag: If $X' \in N_2^c(f(w))$ then there is an $X \subseteq W$ such that

$$f[X] = X' \text{ and } X \in N_1^c(w)$$

Restricting to the non-monotonic core

f is a bounded core morphism from $\mathfrak{M}_1$ to $\mathfrak{M}_2$ provided:

Atomic harmony: for each $p \in \text{At}$, $w \in V(p)$ iff $f(w) \in V'(p)$

Morphism: If $X \in N_1^c(w)$ then $f[X] \in N_2^c(f(w))$

Zag: If $X' \in N_2^c(f(w))$ then there is an $X \subseteq W$ such that

$$f[X] = X' \text{ and } X \in N_1^c(w)$$

Fact: It is not true that f is a bounded core morphism iff $f^{-1}[X] \in N_1^c(w)$ iff $X \in N_1^c(f(w))$.

Restricting to the non-monotonic core

f is a bounded core morphism from $\mathfrak{M}_1$ to $\mathfrak{M}_2$ provided:

Atomic harmony: for each $p \in \mathsf{At}$, $w \in V(p)$ iff $f(w) \in V'(p)$

Morphism: If $X \in N_1^c(w)$ then $f[X] \in N_2^c(f(w))$

Zag: If $X' \in N_2^c(f(w))$ then there is an $X \subseteq W$ such that

$$f[X] = X' \text{ and } X \in N_1^c(w)$$

Proposition If $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are core-complete monotonic models. Then f is a bounded core morphism if f is an injective bounded morphism.

Restricting to the non-monotonic core

$Z \subseteq W \times W'$ is a core bisimulation provided

Atomic harmony: for each $p \in \mathsf{At}$, $w \in V(p)$ iff $w' \in V'(p)$

Zig: If $X \in N^c(w)$ then there is an $X' \subseteq W'$ such that

$$X' \in N'^c(w') \text{ and } \forall x' \in X' \ \exists x \in X \text{ such that } xZx'$$

Zag: If $X' \in N'^c(w')$ then there is an $X \subseteq W$ such that

$$X \in N^c(w) \text{ and } \forall x \in X \ \exists x' \in X' \text{ such that } xZx'$$

Restricting to the non-monotonic core

$Z \subseteq W \times W'$ is a core bisimulation provided

Atomic harmony: for each $p \in \text{At}$, $w \in V(p)$ iff $w' \in V'(p)$

Zig: If $X \in N^c(w)$ then there is an $X' \subseteq W'$ such that

$$X' \in N'^c(w') \text{ and } \forall x' \in X' \; \exists x \in X \text{ such that } xZx'$$

Zag: If $X' \in N'^c(w')$ then there is an $X \subseteq W$ such that

$$X \in N^c(w) \text{ and } \forall x \in X \; \exists x' \in X' \text{ such that } xZx'$$

Lemma: If $\mathfrak{M}$ and $\mathfrak{M}'$ are core-complete models, then Z is a core bisimulation iff Z is a bisimulation.

Restricting to the non-monotonic core

Lemma: Let $\mathfrak{M}$ be a core-complete monotonic models and $\mathfrak{M}'$ a submodel of $\mathfrak{M}$. Then $\mathfrak{M}'$ is a genereated submodel iff

If $w' \in W'$ and $X \in N^c(w')$ then $X \subseteq W'$.

Hennessy-Milner Classes

A neighborhood frame is locally core-finite provided the model is core-complete where each $N^c(w)$ contains finitely many finite sets.

Hennessy-Milner Classes

A neighborhood frame is locally core-finite provided the model is core-complete where each $N^c(w)$ contains finitely many finite sets.

Lemma

If $\mathfrak{M}$ and $\mathfrak{M}'$ are locally core-finite models. Then modal equivalence implies bisimularity.

What about the van Benthem Characterization Theorem?
Goldblatt-Thomason Theorem?

What about the van Benthem Characterization Theorem?
Goldblatt-Thomason Theorem?

Theorem

Let K *be a class of monotonic frames which is closed under taking ultra filter extensions. Then* K *is modally definable iff* K *is closed under disjoint unions, generated subframes, bounded morphic images and it reflects ultrafilter extensions.*

The Language $\mathcal{L}_2$

The language $\mathcal{L}_2$ is built from the following grammar:

$$x = y \mid u = v \mid \mathsf{P}_i x \mid x\mathsf{N}u \mid u\mathsf{E}x \mid \neg\varphi \mid \varphi \wedge \psi \mid \exists x\varphi \mid \exists u\varphi$$

Formulas of $\mathcal{L}_2$ are interpreted in two-sorted first order structures $\mathfrak{M} = \langle D, \{P_i \mid i \in \omega\}, N, E\rangle$ where $D = D^{\mathsf{s}} \cup D^{\mathsf{n}}$ (and $D^{\mathsf{s}} \cap D^{\mathsf{n}} = \emptyset$), each $Q_i \subseteq D^{\mathsf{s}}$, $N \subseteq D^{\mathsf{s}} \times D^{\mathsf{n}}$ and $E \subseteq D^{\mathsf{n}} \times D^{\mathsf{s}}$. The usual definitions of free and bound variables apply.

The Language $\mathcal{L}_2$

Definition

Let $\mathfrak{M} = \langle S, N, V \rangle$ be a neighbourhood model. The *first-order translation* of $\mathcal{M}$ is the structure $\mathfrak{M}^\circ = \langle D, \{P_i \mid i \in \omega\}, R_N, R_\ni \rangle$ where

- $D^\mathsf{s} = S$, $D^\mathsf{n} = \bigcup_{s \in S} N(s)$
- For each $i \in \omega$, $P_i = V(p_i)$
- $R_N = \{(s, U) \mid s \in D^\mathsf{s}, U \in N(s)\}$
- $R_\ni = \{(U, s) \mid s \in D^\mathsf{s}, s \in U\}$

The Language $\mathcal{L}_2$

Definition

Let $\mathfrak{M} = \langle S, N, V \rangle$ be a neighbourhood model. The *first-order translation* of $\mathcal{M}$ is the structure $\mathfrak{M}^\circ = \langle D, \{P_i \mid i \in \omega\}, R_N, R_\ni \rangle$ where

- $D^{\mathsf{s}} = S$, $D^{\mathsf{n}} = \bigcup_{s \in S} N(s)$
- For each $i \in \omega$, $P_i = V(p_i)$
- $R_N = \{(s, U) \mid s \in D^{\mathsf{s}}, U \in N(s)\}$
- $R_\ni = \{(U, s) \mid s \in D^{\mathsf{s}}, s \in U\}$

The Language $\mathcal{L}_2$

Definition

Let $\mathfrak{M} = \langle S, N, V \rangle$ be a neighbourhood model. The *first-order translation* of $\mathcal{M}$ is the structure $\mathfrak{M}^\circ = \langle D, \{P_i \mid i \in \omega\}, R_N, R_\ni \rangle$ where

- $D^{\mathsf{s}} = S$, $D^{\mathsf{n}} = \bigcup_{s \in S} N(s)$
- For each $i \in \omega$, $P_i = V(p_i)$
- $R_N = \{(s, U) \mid s \in D^{\mathsf{s}}, U \in N(s)\}$
- $R_\ni = \{(U, s) \mid s \in D^{\mathsf{s}}, s \in U\}$

The Language $\mathcal{L}_2$

Definition

Let $\mathfrak{M} = \langle S, N, V \rangle$ be a neighbourhood model. The *first-order translation* of $\mathcal{M}$ is the structure $\mathfrak{M}^\circ = \langle D, \{P_i \mid i \in \omega\}, R_N, R_\ni \rangle$ where

- $D^{\mathsf{s}} = S$, $D^{\mathsf{n}} = \bigcup_{s \in S} N(s)$
- For each $i \in \omega$, $P_i = V(p_i)$
- $R_N = \{(s, U) \mid s \in D^{\mathsf{s}}, U \in N(s)\}$
- $R_\ni = \{(U, s) \mid s \in D^{\mathsf{s}}, s \in U\}$

The Language $\mathcal{L}_2$

Definition

Let $\mathfrak{M} = \langle S, N, V \rangle$ be a neighbourhood model. The *first-order translation* of $\mathcal{M}$ is the structure $\mathfrak{M}^\circ = \langle D, \{P_i \mid i \in \omega\}, R_N, R_\ni \rangle$ where

- $D^{\mathsf{s}} = S$, $D^{\mathsf{n}} = \bigcup_{s \in S} N(s)$
- For each $i \in \omega$, $P_i = V(p_i)$
- $R_N = \{(s, U) \mid s \in D^{\mathsf{s}}, U \in N(s)\}$
- $R_\ni = \{(U, s) \mid s \in D^{\mathsf{s}}, s \in U\}$

The Language $\mathcal{L}_2$

Definition

The *standard translation* of the basic modal language are functions $st_x : \mathcal{L} \rightarrow \mathcal{L}_2$ defined as follows as follows: $st_x(p_i) = \mathsf{P}_i x$, st_x commutes with boolean connectives and

$$st_x(\Box\varphi) = \exists u(x\mathsf{R}_N u \wedge (\forall y(u\mathsf{R}_{\ni} y \leftrightarrow st_y(\varphi)))$$

The Language $\mathcal{L}_2$

Definition

The *standard translation* of the basic modal language are functions $st_x : \mathcal{L} \to \mathcal{L}_2$ defined as follows as follows: $st_x(p_i) = \mathsf{P}_i x$, st_x commutes with boolean connectives and

$$st_x(\Box\varphi) = \exists u(x\mathsf{R}_N u \wedge (\forall y(u\mathsf{R}_\ni y \leftrightarrow st_y(\varphi)))$$

Lemma

Let $\mathfrak{M}$ be a neighbourhood structure and $\varphi \in \mathcal{L}$. For each $s \in S$, $\mathfrak{M}, s \models \varphi$ iff $\mathfrak{M}^\circ \models st_x(\varphi)[s]$.

Of course, not every $\mathcal{L}_2$-structures are translations of neighbourhood models.

Of course, not every $\mathcal{L}_2$-structures are translations of neighbourhood models.

$\mathbf{N} = \{\mathfrak{M} \mid \mathfrak{M} \cong \mathfrak{M}^{\circ}$ for some neighbourhood model $\mathfrak{M}\}$

Of course, not every $\mathcal{L}_2$-structures are translations of neighbourhood models.

$\mathbf{N} = \{\mathfrak{M} \mid \mathfrak{M} \cong \mathfrak{M}^\circ$ for some neighbourhood model $\mathfrak{M}\}$

(A1) $\exists x(x = x)$

(A2) $\forall u \exists x(x \mathsf{R}_N u)$

(A3) $\forall u, v(\neg(u = v) \rightarrow$
$\exists x((u\mathsf{R}_\ni x \wedge \neg v\mathsf{R}_\ni x) \vee (\neg u\mathsf{R}_\ni x \wedge v\mathsf{R}_\ni x)))$

Of course, not every $\mathcal{L}_2$-structures are translations of neighbourhood models.

$\mathbf{N} = \{\mathfrak{M} \mid \mathfrak{M} \cong \mathcal{M}^\circ$ for some neighbourhood model $\mathcal{M}\}$

(A1) $\exists x(x = x)$

(A2) $\forall u \exists x(x \mathsf{R}_N u)$

(A3) $\forall u, v(\neg(u = v) \rightarrow$
$\exists x((u \mathsf{R}_\ni x \wedge \neg v \mathsf{R}_\ni x) \vee (\neg u \mathsf{R}_\ni x \wedge v \mathsf{R}_\ni x)))$

Theorem

Suppose $\mathfrak{M}$ is an $\mathcal{L}_2$-structure. Then there is a neighbourhood structure $\mathfrak{M}_\circ$ such that $\mathfrak{M} \cong (\mathfrak{M}_\circ)^\circ$.

$\mathcal{L}_2$ over topological models

Theorem

$\mathcal{L}_2$ interpreted over topological models lacks, Compactness, Löwenheim-Skolem and Interpolation, and is Π^1_1-hard for validity.

B. ten Cate, D. Gabelaia and D. Sustretov. *Modal Language for Topology: Expressivity and Definability*. 2006.

The Language $\mathcal{L}_t$

The language $\mathcal{L}_t$ is a sublanguage of $\mathcal{L}_2$ defined by the following restrictions:

- If α is positive in the open variable u and x is a point variable, then $\forall U(x\mathsf{E}U \rightarrow \alpha)$ is a formula of $\mathcal{L}_t$
- If α is negative in the open variable U and x is a point variable then $\exists U(x\mathsf{E}U \wedge \alpha)$ is a formula of $\mathcal{L}_t$

B. ten Cate, D. Gabelaia and D. Sustretov. *Modal Language for Topology: Expressivity and Definability*. 2006.

The Language $\mathcal{L}_t$

The language $\mathcal{L}_t$ is a sublanguage of $\mathcal{L}_2$ defined by the following restrictions:

- If α is positive in the open variable u and x is a point variable, then $\forall U(x \mathsf{E} U \to \alpha)$ is a formula of $\mathcal{L}_t$
- If α is negative in the open variable U and x is a point variable then $\exists U(x \mathsf{E} U \wedge \alpha)$ is a formula of $\mathcal{L}_t$

Fact: $\mathcal{L}_t$ cannot distinguish between bases and topologies.

B. ten Cate, D. Gabelaia and D. Sustretov. *Modal Language for Topology: Expressivity and Definability*. 2006.

The Language $\mathcal{L}_t$

The language $\mathcal{L}_t$ is a sublanguage of $\mathcal{L}_2$ defined by the following restrictions:

- If α is positive in the open variable u and x is a point variable, then $\forall U(x\mathsf{E}U \to \alpha)$ is a formula of $\mathcal{L}_t$
- If α is negative in the open variable U and x is a point variable then $\exists U(x\mathsf{E}U \wedge \alpha)$ is a formula of $\mathcal{L}_t$

$\mathcal{L}_t$ can express many natural topological properties.

B. ten Cate, D. Gabelaia and D. Sustretov. *Modal Language for Topology: Expressivity and Definability*. 2006.

The Language $\mathcal{L}_t$

The language $\mathcal{L}_t$ is a sublanguage of $\mathcal{L}_2$ defined by the following restrictions:

- If α is positive in the open variable u and x is a point variable, then $\forall U(x \mathsf{E} U \rightarrow \alpha)$ is a formula of $\mathcal{L}_t$
- If α is negative in the open variable U and x is a point variable then $\exists U(x \mathsf{E} U \wedge \alpha)$ is a formula of $\mathcal{L}_t$

$\mathcal{L}_t$ has Compactness, Löwenheim-Skolem and Interpolation.

B. ten Cate, D. Gabelaia and D. Sustretov. *Modal Language for Topology: Expressivity and Definability*. 2006.

Monotonic Fragment of First-Order Logic

On monotonic models:
$st_x^{mon}(\Box\varphi) = \exists u(x\mathsf{R}_N u \wedge (\forall y(u\mathsf{R}_{\ni} y \rightarrow st_y(\varphi)))$.

Monotonic Fragment of First-Order Logic

On monotonic models:
$st_x^{mon}(\Box\varphi) = \exists u(x\mathsf{R}_N u \wedge (\forall y(u\mathsf{R}_{\ni} y \rightarrow st_y(\varphi)))$.

Theorem

A $\mathcal{L}_2$ formula $\alpha(x)$ is invariant for monotonic bisimulation, then $\alpha(x)$ is equivalent to $st_x^{mon}(\varphi)$ for some $\varphi \in \mathcal{L}$.

M. Pauly. *Bisimulation for Non-normal Modal Logic*. 1999.

H. Hansen. *Monotonic Modal Logic*. 2003.

Do monotonic bisimulations work when we drop monotonicity?

Do monotonic bisimulations work when we drop monotonicity? No!

Do monotonic bisimulations work when we drop monotonicity? No!

Can we adapt the definition of bisimulation to the non-monotonic case?

Do monotonic bisimulations work when we drop monotonicity? No!

Can we adapt the definition of bisimulation to the non-monotonic case?

Definition

Two points w_1 from $\mathfrak{F}_1$ and w_2 from $\mathfrak{F}_2$ are behavorially equivalent provided there is a neighborhood frame $\mathfrak{F}$ and bounded morphisms $f : \mathfrak{F}_1 \rightarrow \mathfrak{F}$ and $g : \mathfrak{F}_2 \rightarrow \mathfrak{F}$ such that $f(w_1) = g(w_2)$.

Theorem

Over the class **N** *(of neighborhood models), the following are equivalent:*

- $\alpha(x)$ *is equivalent to the translation of a modal formula*
- $\alpha(x)$ *is invariant under behavioural equivalence.*

H. Hansen, C. Kupke and EP. *Bisimulation for Neighborhood Structures*. CALCO 2007.

What can we infer from the fact that bi-modal normal modal logic can simulate non-normal modal logics?

What can we infer from the fact that bi-modal normal modal logic can simulate non-normal modal logics?

Can we read off a notion of bisimulation?

What can we infer from the fact that bi-modal normal modal logic can simulate non-normal modal logics?

Can we read off a notion of bisimulation? Not clear.

What can we infer from the fact that bi-modal normal modal logic can simulate non-normal modal logics?

Can we read off a notion of bisimulation? Not clear.

- Decidability of the satisfiability problem
- Canonicity
- Salqhvist Theorem
- ????

O. Gasquet and A. Herzig. *From Classical to Normal Modal Logic*. .

M. Kracht and F. Wolter. *Normal Monomodal Logics can Simulate all Others* . .

H. Hansen (Chapter 10). *Monotonic Modal Logics*. 2003.

Theorem The McKinsey Axiom is canonical with respect to neighborhood semantics.

T. Surendonk. *Canonicty for Intensional Logics with Even Axioms.* JSL 2001.

Preview for Tomorrow: Neighborhood Semantics in Action

- Game Logic
- Concurrent PDL
- Common Knowledge
- Coalgebra

Thank You!