Neighborhood Semantics for Modal Logic

Lecture 3

Eric Pacuit

ILLC, Universiteit van Amsterdam
staff.science.uva.nl/~epacuit

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✓ Introduction, Motivation and Background Information

✓ Basic Concepts, Non-normal Modal Logics, Completeness, Incompleteness, Relation with Relational Semantics

Lecture 3: Decidability/Complexity, Related Semantics: Topological Semantics for Modal Logic, More on the Relation with Relational Semantics, Subset Models, First-order Modal Logic

Lecture 4: Advanced Topics — Model Theory

Lecture 5: Neighborhood Semantics in Action: Game Logic, Coalgebra, Common Knowledge, First-Order Modal Logic

Quick Review from Yesterday

Theorem
*The logic **E** is sound and strongly complete with respect to the class of all neighborhood frames.*

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The logic **K** *is sound and strongly complete with respect to the class of filters.*

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The logic **K** *is sound and strongly complete with respect to the class of augmented frames.*

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The logic **E** *is sound and strongly complete with respect to the class of all neighborhood frames.*

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Theorem
The logic **K** *is sound and strongly complete with respect to the class of augmented frames.*

Fact: There are logics that are incomplete with respect to neighborhood semantics.

Recovering Completeness

Definition

A general neighborhood frame is a tuple $\mathbb{F}^g = \langle W, N, \mathcal{A} \rangle$, where $\langle W, N \rangle$ is a neighborhood frame and $\mathcal{A}$ is a collection of subsets of W closed under intersections, complements, and the m_N operator.

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A valuation $V : \mathsf{At} \rightarrow \wp(W)$ is admissible for a general frame $\langle W, N, \mathcal{A} \rangle$ if for each $p \in \mathsf{At}$, $V(p) \in \mathcal{A}$.

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Definition

Suppose that $\mathbb{F}^g = \langle W, N, \mathcal{A} \rangle$ is a general neighborhood frame. A general modal based on $\mathbb{F}^g$ is a tuple $\mathbb{M}^g = \langle W, N, \mathcal{A}, V \rangle$ where V is an admissible valuation.

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Lemma
Let $\mathbb{M}^g$ be an general neighborhood model. Then for each $\varphi \in \mathcal{L}$, $(\varphi)^{\mathbb{M}^g} \in \mathcal{A}$.

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Lemma

Let $\mathbf{L}$ *be any logic extending* $\mathbf{E}$*. Then the general canonical frame validates* $\mathbf{L}$ *(*$\mathbb{F}^g_{\mathbf{L}} \models \mathbf{L}$*).*

Corollary

Any classical modal logic is strongly complete with respect to some class of general frames.

- Decidability
- Comments on Complexity
- Topological Models for Modal Logic
- From Non-Normal Modal Logics to Normal Modal Logics
- Subset Models
- Neighborhood Semantics for First-Order Modal Logic

Filtrations

Let $\mathbb{M} = \langle W, N, V \rangle$ be a neighborhood model and suppose that Σ is a set of sentences from $\mathcal{L}$.

For each $w, v \in W$, we say $w \sim_\Sigma v$ iff for each $\varphi \in \Sigma$, $w \models \varphi$ iff $v \models \varphi$.

For each $w \in W$, let $[w]_\Sigma = \{v \mid w \sim_\Sigma v\}$ be the equivalence class of $\sim_\Sigma$.

If $X \subseteq W$, let $[X]_\Sigma = \{[w] \mid w \in X\}$.

Filtrations

Definition

Let $\mathbb{M} = \langle W, N, V \rangle$ be a neighborhood model and Σ a set of sentences closed under subformulas. A filtration of $\mathbb{M}$ through Σ is a model $\mathbb{M}^f = \langle W^f, N^f, V^f \rangle$ where

1. $W^f = [W]$
2. For each $w \in W$
   2.1 for each $\Box\varphi \in \Sigma$, $(\varphi)^{\mathcal{M}} \in N(w)$ iff $[(\varphi)^{\mathbb{M}}] \in N^f([w])$
3. For each $p \in \mathsf{At}$, $V(p) = [V(p)]$

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Theorem

Suppose that $\mathbb{M}^f = \langle W^f, N^f, V^f \rangle$ is a filtration of $\mathbb{M} = \langle W, N, V \rangle$ through (a subformula closed) set of sentences Σ. Then for each $\varphi \in \Sigma$,

$$\mathbb{M}, w \models \varphi \textit{ iff } \mathbb{M}^f, [w] \models \varphi$$

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3. For each $p \in \text{At}$, $V(p) = [V(p)]$

Corollary

E *has the finite model property. I.e., if* φ *has a model then there is a finite model.*

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Logics with C are in $PSPACE$.

M. Vardi. *On the Complexity of Epistemic Reasoning*. IEEE (1989).

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Can we import results/ideas from model theory for modal logic with respect to Kripke Semantics/Topological Semantics?

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Can we import results/ideas from model theory for modal logic with respect to Kripke Semantics/Topological Semantics?

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Topological Models for Modal Logic

Definition

Topological Space A **topological space** is a neighborhood frame $\langle W, \mathcal{T} \rangle$ where W is a nonempty set and

1. $W \in \mathcal{T}$, $\emptyset \in W$
2. $\mathcal{T}$ is closed under finite intersections
3. $\mathcal{T}$ is closed under arbitrary unions.

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3. $\mathcal{T}$ is closed under arbitrary unions.

A neighborhood of w is any set X such that there is an $O \in \mathcal{T}$ with $w \in O \subseteq N$

Let $\mathcal{T}_w$ be the collection of all neighborhoods of w.

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Lemma

Let $\langle W, \mathcal{T} \rangle$ be a topological space. Then for each $w \in W$, the collection $\mathcal{T}_w$ contains W, is closed under finite intersections and closed under arbitrary unions.

Topological Models for Modal Logic

The largest open subset of X is called the interior of X, denoted $Int(X)$. Formally,

$$Int(X) = \cup\{O \mid O \in \mathcal{T} \text{ and } O \subseteq X\}$$

The smallest closed set containing X is called the closure of X, denoted $Cl(X)$. Formally,

$$Cl(X) = \cap\{C \mid W - C \in \mathcal{T} \text{ and } X \subseteq C\}$$

Topological Models for Modal Logic

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Lemma

Let $\langle W, \mathcal{T}\rangle$ be a topological space and $X \subseteq W$. Then

1. $Int(X \cap Y) = Int(X) \cap Int(Y)$
2. $Int(\emptyset) = \emptyset$, $Int(W) = W$
3. $Int(X) \subseteq X$
4. $Int(Int(X)) = Int(X)$
5. $Int(X) = W - Cl(W - X)$

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Lemma

Let $\langle W, \mathcal{T}\rangle$ be a topological space and $X \subseteq W$. Then

1. $\Box(\varphi \wedge \psi) \leftrightarrow \Box\varphi \wedge \Box\psi$
2. $\Box\bot \leftrightarrow \bot, \Box\top \leftrightarrow \top$
3. $\Box\varphi \rightarrow \varphi$
4. $\Box\Box\varphi \leftrightarrow \Box\varphi$
5. $\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$

Topological Models for Modal Logic

A topological model is a triple $\langle W, \mathcal{T}, V\rangle$ where $\langle W, \mathcal{T}\rangle$ is a topological space and V a valuation function.

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$$\mathbb{M}^T, w \models \Box\varphi \text{ iff } \exists O \in \mathcal{T}, w \in O \text{ such that } \forall v \in O, \mathbb{M}^T, v \models \varphi$$

$$(\Box\varphi)^{\mathbb{M}^T} = Int((\varphi)^{\mathbb{M}^T})$$

From Neighborhoods to Topologies

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Fact: A family $\mathcal{B}$ of subsets of W is a basis for some topology if

- for each $w \in W$ there is a $U \in \mathcal{B}$ such that $w \in U$
- for each $U, V \in \mathcal{B}$, if $w \in U \cap V$ then there is a $W \in \mathcal{B}$ such that $w \in W \subseteq U \cap V$

From Neighborhoods to Topologies

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Let $\mathbb{M} = \langle W, N, V \rangle$ be a neighborhood models. Suppose that N satisfies the following properties

- for each $w \in W$, $N(w)$ is a filter
- for each $w \in W$, $w \in \cap N(w)$
- for each $w \in W$ and $X \subseteq W$, if $X \in N(w)$, then $m_N(X) \in N(w)$

Then there is a topological model that is point-wise equivalent to $\mathbb{M}$.

Main Completeness Result

Theorem

S4 *is the logic of the class of all topological spaces.*

J. van Benthem and G. Bezhanishvili. *Modal Logics of Space*. Handbook of Spatial Logics (2007).

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We can *simulate* any non-normal modal logic with a bi-modal normal modal logic.

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Given a neighborhood model $\mathcal{M} = \langle W, N, V \rangle$, define a Kripke model $\mathcal{M}^\circ = \langle V, R_N, R_{\not\ni}, R_N, Pt, V \rangle$ as follows:

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- $R_N = \{(w, u) \mid w \in W, u \in \wp(W), u \in N(w)\}$

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- $Pt = W$

Let $\mathcal{L}'$ be the language

$$\varphi := p \mid \neg\varphi \mid \varphi \wedge \psi \mid [\ni]\varphi \mid [\not\ni]\varphi \mid [N]\varphi \mid \text{Pt}$$

where $p \in$ At and Pt is a unary modal operator.

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Lemma

For each neighborhood model $\mathcal{M} = \langle W, N, V \rangle$ and each formula $\varphi \in \mathcal{L}$, for any $w \in W$,

$$\mathcal{M}, w \models \varphi \text{ iff } \mathcal{M}^\circ, w \models ST(\varphi)$$

Monotonic Models

Lemma

On Monotonic Models $\langle N\rangle([\ni]ST(\varphi) \wedge [\not\ni]\neg ST(\varphi))$ *is equivalent to* $\langle N\rangle([\ni]ST(\varphi))$

More on this tomorrow!

O. Gasquet and A. Herzig. *From Classical to Normal Modal Logic*. .

M. Kracht and F. Wolter. *Normal Monomodal Logics can Simulate all Others*.
.

The key idea is to replace neighborhood models with a two-sorted Kripke model.

- Decidability
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A Logic for Two-sorted Neighborhood Structures

A. Dabrowski, L. Moss and R. Parikh. *Topological Reasoning and The Logic of Knowledge*. APAL (1996).

R. Parikh, L. Moss and C. Steinsvold. *Topology and Epistemic Logic*. Handbook of Spatial Logic (2007).

Subset Models

A Subset Frame is a pair $\langle W, \mathcal{O} \rangle$ where

- W is a set of states
- $\mathcal{O} \subseteq \wp(W)$ is a set of subsets of W, i.e., a set of *observations*

Neighborhood Situation: Given a subset frame $\langle W, \mathcal{O} \rangle$, (w, U) is called a neighborhood situation, provided $w \in U$ and $U \in \mathcal{O}$.

Model: $\langle W, \mathcal{O}, V \rangle$, where $V : \mathsf{At} \rightarrow \wp(W)$ is a valuation function.

Language:

$$\varphi := p \mid \varphi \wedge \varphi \mid \neg \varphi \mid K\varphi \mid \Diamond \varphi$$

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- $w, U \models K\varphi$ iff for all $v \in U$, $v, U \models \varphi$
- $w, U \models \Diamond\varphi$ iff there is a $V \in \mathcal{O}$ such that $w \in V$ and $w, V \models \varphi$

Axioms

1. All propositional tautologies
2. $(p \rightarrow \Box p) \wedge (\neg p \rightarrow \Box \neg p)$, for $p \in \mathsf{At}$.
3. $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$
4. $\Box\varphi \rightarrow \varphi$
5. $\Box\varphi \rightarrow \Box\Box\varphi$
6. $K(\varphi \rightarrow \psi) \rightarrow (K\varphi \rightarrow K\psi)$
7. $K\varphi \rightarrow \varphi$
8. $K\varphi \rightarrow KK\varphi$
9. $\neg K\varphi \rightarrow K\neg K\varphi$
10. $K\Box\varphi \rightarrow \Box K\varphi$

We include the following rules: modus ponens, K_i-necessitation and $\Box$-necessitation.

Theorem

The previous axioms are sound and complete for the class of all subset models.

L. Moss and R. Parikh. *Topological Reasoning and The Logic of Knowledge*. TARK (1992).

Fact: $\Box\Diamond\varphi \to \Diamond\Box\varphi$ is sound for spaces closed under intersections.

Fact: $\Diamond\varphi \wedge L\Diamond\psi \to \Diamond[\Diamond\varphi \wedge L\Diamond\psi \wedge K\Diamond L(\varphi \vee \psi)]$ is sound for spaces closed under binary unions.

Overview of Results

- (Georgatos: 1993, 1994, 1997) completely axiomatized Topologic where $\mathcal{O}$ is restricted to a topology and showed that the logic has the finite model property. Similarly for treelike spaces.
- (Weiss and Parikh: 2002) showed that an infinite number of axiom schemes is required to axiomatize Topologics in which $\mathcal{O}$ is closed under intersection.
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A formula of *first-order modal logic* will have the following syntactic form

$$\varphi := F(x_1, \ldots, x_n) \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Box\varphi \mid \forall x\varphi$$

A **constant domain neighborhood frame** is a tuple $\langle W, N, D\rangle$ where W and D are sets, and $N : W \to \wp\wp(W)$.

A **constant domain neighborhood model** is a tuple $\langle W, N, D, I\rangle$, where for each n-ary relation symbol F and $w \in W$, $I(F, w) \subseteq D^n$.

A **substitution** is any function $\sigma : \mathcal{V} \to D$.

A substitution σ' is said to be an x-**variant** of σ if $\sigma(y) = \sigma'(y)$ for all variable y except possibly x, this will be denoted by $\sigma \sim_x \sigma'$.

Let $\mathcal{M} = \langle W, N, D, I\rangle$ be any constant domain neighborhood model and σ any substitution

1. $\mathcal{M}, w \models_\sigma F(x_1, \ldots, x_n)$ iff $\langle \sigma(x_1), \ldots, \sigma(x_n)\rangle \in I(F, w)$
2. $\mathcal{M}, w \models_\sigma \Box\varphi$ iff $(\varphi)^{\mathcal{M},\sigma} \in N(w)$
3. $\mathcal{M}, w \models_\sigma \forall x\varphi(x)$ iff for each x-variant σ', $\mathcal{M}, w \models_{\sigma'} \varphi(x)$

Classical First-order Modal Logic

Let **S** be any classical propositional modal logic, by **FOL** + **S** we mean the set of formulas closed under the following rules and axiom schemes:

S All axiom schemes and rules from **S**.

∀ $\forall x\varphi(x) \rightarrow \varphi[y/x]$ is an axiom scheme.

Gen $\dfrac{\varphi \rightarrow \psi}{\varphi \rightarrow \forall x\psi}$, where x is not free in φ.

Barcan Schemas

- **Barcan formula** (BF): $\forall x \Box \varphi(x) \rightarrow \Box \forall x \varphi(x)$
- **converse Barcan formula** (CBF): $\Box \forall x \varphi(x) \rightarrow \forall x \Box \varphi(x)$

Barcan Schemas

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Observation 3: BF is valid in a *varying* domain relational frame iff the frame is anti-monotonic; CBF is valid in a *varying* domain relational frame iff the frame is monotonic.

Fitting and Mendelsohn. *First-Order Modal Logic*. 1998.

High Probability

The *BF* instantiates cases of what is usually known as the '**lottery paradox**':

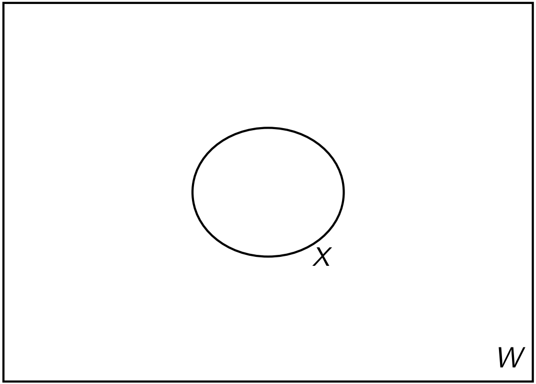
For each individual x, it is *highly probably* that x will loose the lottery; however it is not necessarily highly probably that each individual will loose the lottery.

Converse Barcan Formulas and Neighborhood Frames

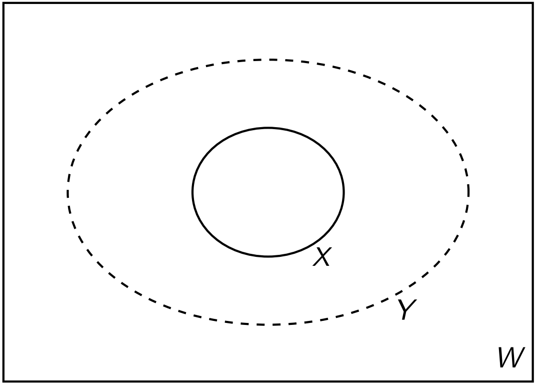
A frame $\mathcal{F}$ is **consistent** iff for each $w \in W$, $N(w) \neq \emptyset$

A first-order neighborhood frame $\mathcal{F} = \langle W, N, D \rangle$ is **nontrivial** iff $|D| > 1$

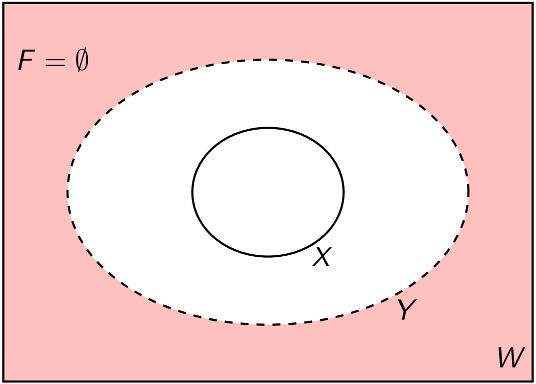
Lemma Let $\mathcal{F}$ be a consistent constant domain neighborhood frame. The converse Barcan formula is valid on $\mathcal{F}$ iff either $\mathcal{F}$ is trivial or $\mathcal{F}$ is supplemented.



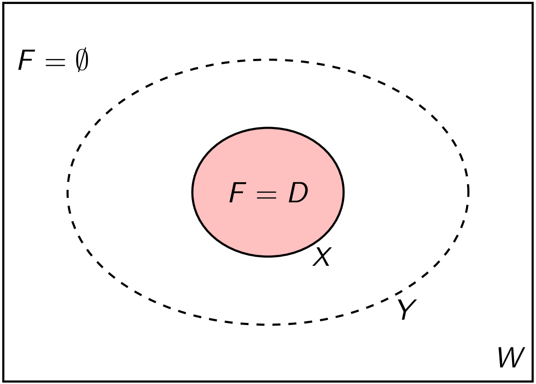
$$X \in N(w)$$



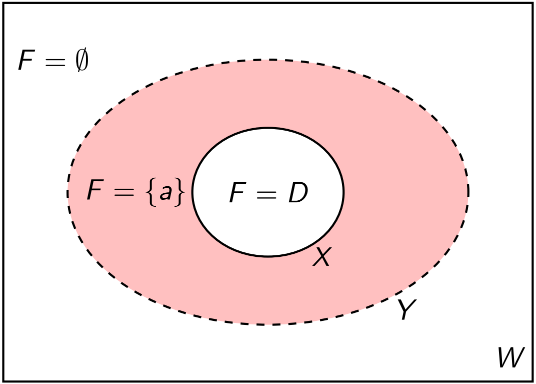
$$Y \notin N(w)$$



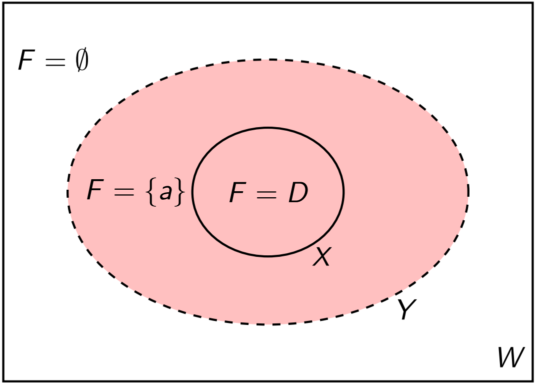
$$\forall v \notin Y, \quad I(F, v) = \emptyset$$



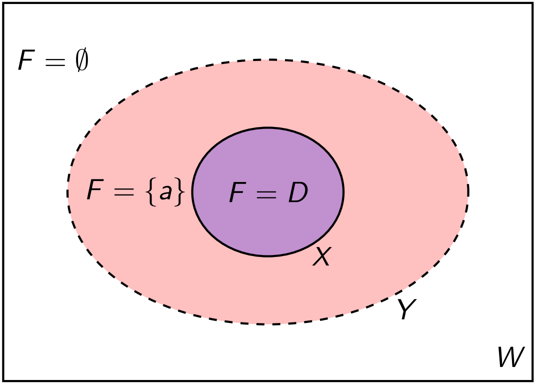
$$\forall v \in X, \quad I(F, v) = D = \{a, b\}$$



$$\forall v \in Y - X, \quad I(F, v) = \{a\}$$



$$(F[a])^{\mathcal{M}} = Y \not\in N(w) \quad \text{hence} \quad w \not\models \forall x \Box F(x)$$



$$(\forall x F(x))^{\mathcal{M}} = (F[a])^{\mathcal{M}} \cap (F[b])^{\mathcal{M}} = X \in N(w) \quad \text{hence} \quad w \models \Box\forall x F(x)$$

Barcan Formulas and Neighborhood Frames

We say that a frame closed under $\leq \kappa$ intersections if for each state w and each collection of sets $\{X_i \mid i \in I\}$ where $|I| \leq \kappa$, $\cap_{i \in I} X_i \in N(w)$.

Lemma Let $\mathcal{F}$ be a consistent constant domain neighborhood frame. The Barcan formula is valid on $\mathcal{F}$ iff either

1. $\mathcal{F}$ is trivial or
2. if D is finite, then $\mathcal{F}$ is closed under finite intersections and if D is infinite and of cardinality κ, then $\mathcal{F}$ is closed under $\leq \kappa$ intersections.

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Theorem $\mathbf{FOL} + \mathbf{E} + CBF$ is sound and strongly complete with respect to the class of frames that are either non-trivial and supplemented or trivial and not supplemented.

FOL $+$ **K** and **FOL** $+$ **K** $+$ *BF*

Theorem **FOL** $+$ **K** is sound and strongly complete with respect to the class of filters.

FOL + K and FOL + K + *BF*

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Observation The augmentation of the smallest canonical model for **FOL** + **K** is not a canonical model for **FOL** + **K**. In fact, the closure under infinite intersection of the minimal canonical model for **FOL** + **K** is not a canonical model for **FOL** + **K**.

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Lemma The augmentation of the smallest canonical model for **FOL** + **K** + *BF* is a canonical for **FOL** + **K** + *BF*.

Theorem **FOL** + **K** + *BF* is sound and strongly complete with respect to the class of augmented first-order neighborhood frames.

H. Arlo-Costa and EP. *Classical Systems of First-Order Modal Logic*. Studia Logica (2006).

Thank You!